Part-II

Analytical Dynamics

M.Sc Math

Analytical Dynamics

New Notes

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Analytical Dynamics :-

Dynamics:-

Every thing in motion is called dynamics.

System:-

The object on which we focus our attention is called system.

Dynamical System:-

A dynamical system consist of n particles such that every particle have its own identification i.e. every particle has certain mass and we can trace the position of each particle at any time.

Constraint:-

Any thing which limits the motion of a dynamical system is called constraint.

or

Constraint is a sort of restriction which is imposed on a dynamical system.

Examples of Constraints:-

i) Gas molecules contained in a cylinder are constraint by the walls of the cylinder, to move only inside the cylinder.

2. Rigid body as a collection of particles (which are kept at invariant distances) is an example of constraint. In this case the condition of constraint demands that,

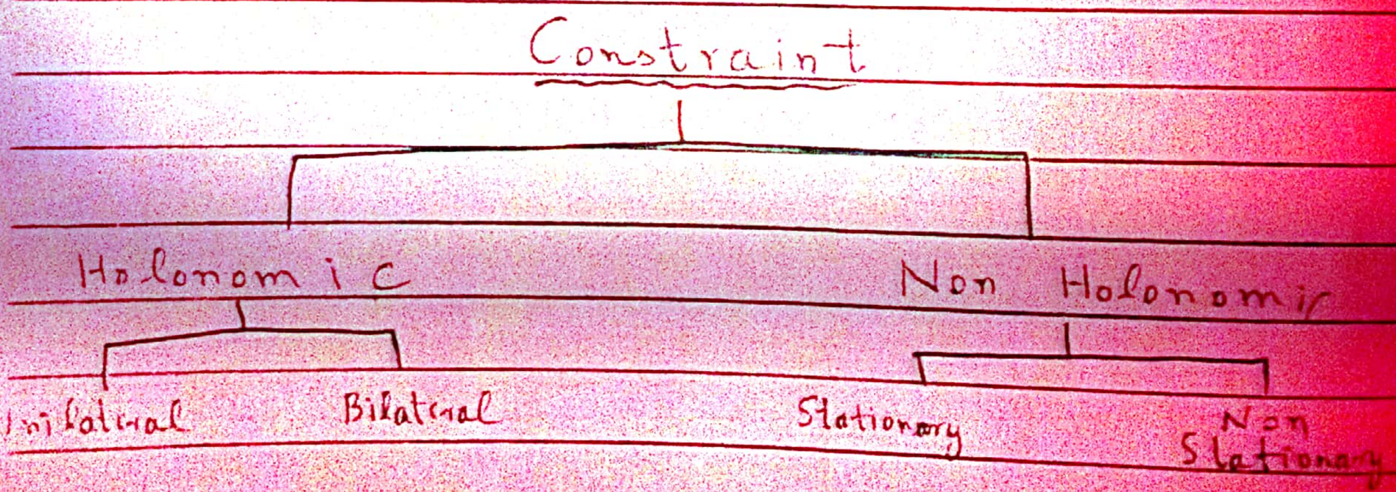
$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

is invariant.

Where, $P_1(x_1, y_1, z_1)$ & $P_2(x_2, y_2, z_2)$ are any two points of rigid body.

-: Classification of Constraints:-

The constraints involve in a dynamical system can be classified as follows.



These Notes are Lectures Deliverd by Sir Saqib.

Holonomic Constraint:-

If the condition of constraints are expressed by relations connecting the coordinates and possibly time then these constraints are said to be Holonomic constraints.

If there are N particles in a dynamical system and r constraints are imposed on the system. Suppose all the constraints are holonomic then the relation is of the form,

$$f_l(x_1, x_2, \dots x_{3N}, t) = 0 \quad \text{—— (1)}$$

$$l = 1, 2, \dots r$$

Here eq. (1) will represent "r" relations where $r \not> 3N$ but $r < 3N$ because even if $r = 3N$ then motion of the system is determine by the equation of constraint. Thus we can find all the 3N coordinates by solving 3N equations simultaneously which means that the entire motion of system is due to constraint. This is a contradiction because the constraints are restriction on the motion.

The holonomic constraints can also be expressed as,

$$f_l(x_1, x_2, \ldots x_N, t) < 0 \quad \text{—— (2)}$$

where $l = 1, 2, \ldots r$

The holonomic constraints are bilateral if the relations representing the constraints are equations and unilateral if the relations are inequalities.

Examples of holonomic constraints:-

(i) Rigid body (ii) the particle moving in a circle with fixed radius

Non Holonomic Constraints :-

The constraints are said to be non holonomic if the coordinates and the velocities of the system are connected by:

$$\phi_l(x_1, x_2, \ldots x_{3N}, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_{3N}, t) = 0$$

where, $l = 1, 2, 3, \ldots r$

Where ϕ_l are non linear in $\dot{x}_\nu$, $\nu = 1, 2, \ldots 3N$ (In general)

The case which is most important the one for which ϕ_l are linear in $\dot{x}_\nu$. The limitation may be expressed as

$$A_{l\nu}\dot{x}_\nu + A_l = 0 \quad \text{—— (*)}$$

where, $l = 1, 2, 3, \ldots r$

& $\nu = 1, 2, 3, \ldots 3N$

also $A_{l\nu}$ are the functions of $x_1, x_2, \ldots x_{3N}$ & t.

If the time "t" is explicitly involved

Non :: $\frac{dx}{dt} + x + t = 0$ (explicitly) $\quad \frac{}{dt}$ ... Stationary (J).

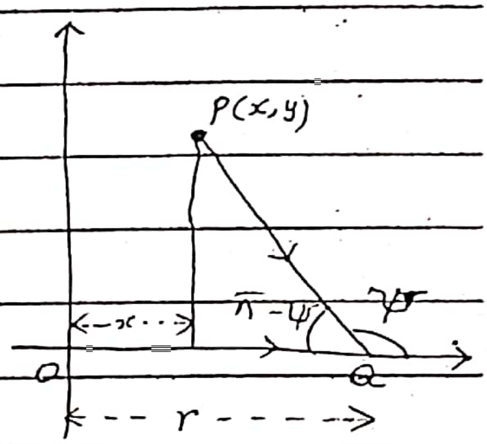
in the eq. (*) then the constraints are said to be non stationary otherwise stationary

Note:-

For stationary non holonomic constraints eq. (*) can be written as,

$$A_{\ell\nu}\,\dot{x}_\nu = 0$$

Example of non holonomic constraint:-



Consider a particle "Q" is moving on x-axis such that the distance OQ is "r" (& r = r(t)). A particle P whose coordinates at any time "t" are given by (x, y) is moving in xy plane in such a way that its velocity at each instant is directed toward Q.

From figure,

$$\tan(\pi - \psi) = \frac{y}{r - x}$$

$$\Rightarrow \quad -\tan\psi = \frac{y}{r - x}$$

But, $\tan\psi = \frac{dy}{dx}$

Therefore, $-\frac{dy}{dx} = \frac{y}{r - x}$

or $\dfrac{dy}{dx} = \dfrac{y}{x-r}$

This is a non integrable equation so is an example of non holonomic constraints.

$\Rightarrow \dfrac{dy/dt}{dx/dt} = \dfrac{y}{x-r} \Rightarrow \dfrac{\dot{y}}{\dot{x}} = \dfrac{y}{x-r}$

$(x-r)\,\dot{y} = y\,\dot{x}$

Holonomic Dynamical System:- A system subject to holonomic constraints is said to be holonomic dynamical system.

Non Holonomic Dynamical System:- If ~~all~~ constraints imposed on a dynamical system are non holonomic then the system is said to be non holonomic dynamical system.

Q. Define holonomic and non holonomic dynamical systems, also give examples in each case.

An

-: Degree of freedom :-

The number of independent coordinates required to determine the configuration of a dynamical system is called its degree of freedom.

Examples.

1, Let a particle is restricted to move on the surface of sphere centered at origin and radius 'a'. Then eq. of motion is given by,

$$x^2 + y^2 + z^2 = a^2$$

or $$z = \pm \sqrt{a^2 - x^2 - y^2} = f(x, y)$$

Since number of independent coordinates require to determine the position of particle is '2' therefore this system has "2" degree of freedom.

2. Consider a particle moving in space under the following restrictions

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

Since, No. of coordinates involved = 3
& " " eq.s = 2
So No. of independent coordinates = 3 - 2 = 1
So, system has only '1' degree of freedom.

Generalized Coordinates:-

The smallest possible number of coordinates of a dynamical system such that,

(i) These values determine the configuration of the system.

(ii) They may be varied orbitarly and independently without violating any constraint of the system are called generalized coordinates.

A dynamical system of N particles free from constraints has 3N degree of freedom. Suppose $x_1, x_2, x_3, \ldots x_{3N}$ be the coordinates required to determine the configuration of system.

Suppose there exist holonomic constraints on this system. Let the equation of constraint be,

$$f_l(x_1, x_2, \ldots x_{3N}, t) = 0 \quad \text{—— (1)}$$

$$l = 1, 2, \ldots r \quad , \quad r < 3N$$

We may use these equations to eliminate "r" coordinates out of 3N coordinates. Therefore we are left with 3N − r independent coordinates. In fact we express r coordinates from eq. (1) in terms of 3N − r independent

coordinates. The relations may be expressed as,

$$\left.\begin{aligned} x_1 &= x_1(x_{r+1}, x_{r+2}, \dots x_{3N}, t) \\ x_2 &= x_2(x_{r+1}, x_{r+2}, \dots x_{3N}, t) \\ &\dots \\ x_r &= x_r(x_{r+1}, x_{r+2}, \dots x_{3N}, t) \end{aligned}\right\} \quad (2)$$

The elimination of dependent coordinates may be expressed in another way by introducing $3N-r$ independent coordinates so relation (2) can be written as,

$$\left.\begin{aligned} x_1 &= x_1(q_1, q_2, \dots q_{3N-r}, t) \\ x_2 &= x_2(q_1, q_2, \dots q_{3N-r}, t) \\ &\dots \\ x_r &= x_r(q_1, q_2, \dots, q_{3N-r}, t) \end{aligned}\right\} \quad (3)$$

These are called coordinate transformations and $q_1, q_2, \dots q_{3N-r}$ are called generalized coordinates.

Usually we put,

$$3N - r = n$$

Configuration Space:-

The space determined by generalized coordinates is called configuration space.

Genralized Velocities :-

The time derivatives of genralized coordinates are called generalized velocities.

If $q_1, q_2, — q_n$ are the generalized coordinates then $\dot{q}_1, \dot{q}_2, -- \dot{q}_n$ are called genralized velocities.

Let us consider a system of N particles whose configuration at any time "t" is specified by n generalized coordinates $q_1, q_2, \dots, q_n$.

Let r_i be the position vector of ith particle.

Therefore,

$$r_i = r_i(q_1; q_2, -- q_n; t) \quad —— (1)$$

$$i = 1, 2, -- N.$$

Diff. eq. (1) w.r.t "t"

$$\frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt} + ---$$

$$---+ \frac{\partial r_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}$$

$$= \frac{\partial r_i}{\partial q_1}\dot{q}_1 + \frac{\partial r_i}{\partial q_2}\dot{q}_2 + --- + \frac{\partial r_i}{\partial q_n}\dot{q}_n + \frac{\partial r_1}{\partial t}$$

$$\dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}$$

or

$$\dot{r}_i = \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \quad —— (2)$$

$x = g(t) \quad , \quad y = h(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad //$$

If time "t" is not involved explictly then eq ② can be written as,

$$\boxed{\dot{r}_i = \frac{\partial r_i}{\partial q_j}\dot{q}_j \; \cancel{+\frac{\partial r_i}{\partial t}}} \quad \text{(Einstein Summation Convention)} \quad —— ②$$

Generalized Accelaration:-

Let us consider a dynamical system of N particles whose configuration at any time "t" is specified by n generalized coordinates $q_1, q_2, \ldots q_n$

Let "r_i" be the position vector of ith particle then,

$$r_i = r_i(q_1, q_2, \ldots q_n, t) \quad —— ①$$

Velocity of ith particle is given by

$$\dot{r}_i = \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \quad \longrightarrow ②$$

Diff. eq ② w.r.t "t"

$$\ddot{r}_i = \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right)$$

$$= \frac{\partial r_i}{\partial q_j}\ddot{q}_j + \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)\dot{q}_j + \frac{d}{dt}\left(\frac{\partial r_i}{\partial t}\right)$$

$$= \frac{\partial r_i}{\partial q_j}\ddot{q}_j + \left[\frac{\partial}{\partial q_k}\left(\frac{\partial r_i}{\partial q_j}\right)\dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial r_i}{\partial q_j}\right)\right]\dot{q}_j$$

$$+ \frac{\partial}{\partial q_k}\left(\frac{\partial r_i}{\partial t}\right)\dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial r_i}{\partial t}\right) \quad \text{using ②}$$

$$\ddot{r}_i = \frac{\partial r_i}{\partial q_j}\ddot{q}_j + \frac{\partial^2 r_i}{\partial q_k \partial q_j}\dot{q}_k\dot{q}_j + \frac{\partial^2 r_i}{\partial t \partial q_j}\dot{q}_j$$

$$+ \frac{\partial^2 r_i}{\partial q_k \partial t}\dot{q}_k + \frac{\partial^2 r_i}{\partial t^2}$$

$$= \frac{\partial r_i}{\partial q_j}\ddot{q}_j + \frac{\partial}{\partial q_j}\left\{\frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}\right\}\dot{q}_j$$

$$+ \frac{\partial}{\partial t}\left\{\frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}\right\}$$

Using eq. (2) we obtain,

$$\ddot{r}_i = \frac{\partial r_i}{\partial q_j}\ddot{q}_j + \frac{\partial}{\partial q_j}(\dot{r}_i)\dot{q}_j + \frac{\partial}{\partial t}(\dot{r}_i)$$

Which is the required expression for generalized acceleration.

Kinetic Energy of a dynamical system in terms of generalized -: coordinates:-

Let us consider a dynamical system of "N particles whose configuration at any time "t" is specified by n generalized coordinates $q_1, q_2, \ldots q_n$. Let m_i be the mass and r_i be the position vector of ith particle at any time 't'.

then, $r_i = r_i(q_1, q_2, \ldots q_n, t)$ —(1)

Also we know ~~if~~ that,

$$\dot{r}_i = \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \quad \text{—(2)}$$

Let "T_i" be the K.E of ith particle then

$$T_i = \frac{1}{2} m_i (v_i)^2$$

The K.E. of the whole system is given by,

$$T = \sum_{i=1}^{N} T_i$$

$$\Rightarrow T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{r}_i^2$$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{r}_i^2 \quad \text{—(3)}$$

Now $\dot{r}_i^2 = \left(\frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right)\left(\frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}\right)$

$$= \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_k}\dot{q}_j\dot{q}_k + \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial t}\dot{q}_j + \frac{\partial r_i}{\partial t}\frac{\partial r_i}{\partial q_k}\dot{q}_k + \left(\frac{\partial r_i}{\partial t}\right)^2$$

$$\dot{r}_i^2 = \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_k}\dot{q}_j\dot{q}_k + 2\frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial t}\dot{q}_j + \left(\frac{\partial r_i}{\partial t}\right)^2$$

Put in (3)

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \left\{\frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_k}\dot{q}_j\dot{q}_k + 2\frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial t}\dot{q}_j + \left(\frac{\partial r_i}{\partial t}\right)^2\right\}$$

Prip. of virtual work   S = Virtual disp.
$F_i \, \delta r_i = 0$ d = Real disp.

Outer product
$A_i \times B_i \times e_p = C_{ep}$

$$= a_2 \dot{q}_j \dot{q}_k + a_1 \dot{q}_j + a_0$$

where,
$$a_2 = \frac{1}{2} \sum_{i=1}^{N} m_i \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k}$$

$$a_1 = \sum_{i=1}^{N} m_i \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial t}$$

$$a_0 = \frac{1}{2} \sum_{i=1}^{N} m_i \left(\frac{\partial r_i}{\partial t}\right)^2$$

:- D' Alembert Principle :-

Let us consider a dynamical system of N particles whose configuration at any time "t" is specified by, $q_1, q_2, \dots q_n$ where q's are generalized coordinates. The position vector of ith particle is given by,

$$r_i = r_i(q_1, q_2, \dots q_n, t) \quad \text{—} \; (1)$$

Let F_i be the total external force applied on ith particle then by principle of virtual work,

$$F_i \, \delta r_i = 0 \quad \text{—} \; (2)$$

Where δ_{r_i} is the virtual disp.
From eq (1)

$$\delta r_i = \frac{\partial r_i}{\partial q_j} \delta q_j$$

So from (2)

$$F_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j = 0$$

$F = ma$

$\frac{dF}{dt} = m\frac{dv}{dt}$

$\frac{dF}{dt} = \dot{p}$

$$\Rightarrow \quad Q_j \delta q_j = 0$$

Where, $Q_j = F_i \cdot \frac{\partial r_i}{\partial q_j}$ is called the generalized force.

By Newton's second law of motion,

$$F_i = \dot{P}_i$$

where P_i is the momentum of ith particle

$$F_i - \dot{P}_i = 0$$

$$\Rightarrow \quad F_i - m_i \ddot{r}_i = 0$$

$$(F_i - m_i \ddot{r}_i)\, \delta r_i = 0$$

This expression is known as D Alempert's princile.

-: Lagrange Eq. of motion:-

Let us consider a dynamical system of "N" particles whose configuration at any time "t" is specified by $q_1, q_2, - q_n$. Let r_i be the p.v. and m_i be the mass of ith particle.

Therefori, $r_i = r_i(q_1, q_2, -- q_n, t)$ — ①

Also,

$$\dot{r}_i = \frac{\partial r_i}{\partial q_s}\dot{q}_s + \frac{\partial r_i}{\partial t} \quad \text{—— ②}$$

Diff eq ② partily w.r.t $\dot{q}_s$

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_s} = \frac{\partial r_i}{\partial q_s}(1) + 0 \quad \text{—— ③}$$

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_s} = \frac{\partial r_i}{\partial q_s}$$

By D, Alembert's principle

$$(F_i - m_i\ddot{r}_i)\delta r_i = 0 \quad \text{—— ④}$$

From ①

$$\delta r_i = \frac{\partial r_i}{\partial q_s}\delta q_s$$

Put in ④

$$[F_i - m_i\ddot{r}_i]\frac{\partial r_i}{\partial q_s}\delta q_s = 0$$

$$\left[F_i\frac{\partial r_i}{\partial q_s} - m_i\ddot{r}_i\frac{\partial r_i}{\partial q_s}\right]\delta q_s = 0 \quad \text{—— ⑤}$$

Consider

$$\frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_s}\right) = \ddot{r}_i \frac{\partial r_i}{\partial q_s} + \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_s}\right)$$

$$= \ddot{r}_i \frac{\partial r_i}{\partial q_s} + \dot{r}_i \left\{\frac{\partial}{\partial q_j}\left(\frac{\partial r_i}{\partial q_s}\right)\dot{q}_j + \frac{\partial}{\partial t}\left(\frac{\partial r_i}{\partial q_s}\right)\right\}$$

$$= \ddot{r}_i \frac{\partial r_i}{\partial q_s} + \dot{r}_i \left\{\frac{\partial^2 r_i}{\partial q_j \partial q_s}\dot{q}_j + \frac{\partial^2 r_i}{\partial t \partial q_s}\right\}$$

$$= \ddot{r}_i \frac{\partial r_i}{\partial q_s} + \dot{r}_i \frac{\partial}{\partial q_s}\left\{\frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right\}$$

$$\Rightarrow \frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_s}\right) = \ddot{r}_i \frac{\partial r_i}{\partial q_s} + \dot{r}_i \frac{\partial}{\partial q_s}\dot{r}_i \quad \text{from (2)}$$

$$\ddot{r}_i \frac{\partial r_i}{\partial q_s} = \frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_s}\right) - \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_s} \quad \text{--- (6)}$$

Now consider,

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_s}\right) = \frac{\partial}{\partial q_j}\left(\frac{\partial r_i}{\partial q_s}\right)\dot{q}_j + \frac{\partial}{\partial t}\left(\frac{\partial r_i}{\partial q_s}\right)$$

$$= \frac{\partial}{\partial q_s}\left\{\frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right\}$$

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_s}\right) = \frac{\partial}{\partial q_s}(\dot{r}_i) \quad \text{--- (*)}$$

Therefore eq. (6) becomes,

$$\ddot{r}_i \frac{\partial r_i}{\partial q_s} = \frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_s}\right) - \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_s}\right)$$

$$\Rightarrow m_i \ddot{r}_i \frac{\partial r_i}{\partial q_s} = \frac{d}{dt}\left(m_i \dot{r}_i \frac{\partial r_i}{\partial q_s}\right) - m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_s}\right) \quad \text{--- (7)}$$

Now as,

$r_i = q_1 q_2$

$\dot{r}_i = q_1 \dot{q}_2 + \dot{q}_1 q_2$

$$T = \frac{1}{2} m_i \dot{r}_i^2$$

$$\Rightarrow \frac{\partial T}{\partial q_s} = \frac{1}{2} m_i \left(2 \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_s} \right)$$

$$= m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_s}$$

$$\frac{\partial T}{\partial q_s} = m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_s} \right) \quad \text{—— (8)}$$

Diff. T w.r.t $\dot{q}_s$

$$\frac{\partial T}{\partial \dot{q}_s} = \frac{1}{2} m_i 2\dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_s}$$

$$\frac{\partial T}{\partial \dot{q}_s} = m_i \dot{r}_i \frac{\partial r_i}{\partial q_s} \quad \text{using (3)} \quad \text{—— (9)}$$

Put eq. (8) & (9) in (7)

$$m_i \ddot{r}_i \frac{\partial r_i}{\partial q_s} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s} \right) - \frac{\partial T}{\partial q_s}$$

Put in (5)

$$\left[F_i \frac{\partial r_i}{\partial q_s} - \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s} \right) - \frac{\partial T}{\partial q_s} \right\} \right] \delta q_s = 0$$

$$\Rightarrow \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s} \right) - \frac{\partial T}{\partial q_s} - Q_s \right\} \delta q_s = 0 \quad \text{—— (10)}$$

where, $Q_s = F_i \frac{\partial r_i}{\partial q_s}$ is generalized force

The expression (10) is known as general equation of dynamics.

ii) Work done along closed path is zero

Q. Derive general eq. of dynamics

or

Derive fundamental eq. of dynamics

or

Derive Langrange's eq. of motion for non conservative system.

Langrange's Eq.s of motion for a

-: conservative Systems :-

A force F is said to be conservative if it is equal to negative of potential gradient. (Gradient of P.E.) $F = -\frac{\partial V}{\partial q_s} = -\left(\frac{\partial V}{\partial q}\right)$

When a system is conservative, the force can be derived from potential energy "V" as follows,

$$Q_s = \frac{-\partial V}{\partial q_s}$$

Put in eq. (10)

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s}\right] \delta q_s = 0$$

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial}{\partial q_s}(T-V)\right] \delta q_s = 0 \quad \text{--- } (*)$$

Let,

$$L = T - V$$

Therefore,

As, $L = L(q_s, \dot{q}_s, t)$

$$\frac{\partial L}{\partial \dot{q}_s} = \frac{\partial T}{\partial \dot{q}_s}$$

Therefore (*) becomes,

$$\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s}\right\} \delta q_s = 0 \quad \text{--- (11)}$$

Eq. (11) is known as Lagrange's eq. of motion for conservative system. and "L" is called Langrangian function or Langrangian.

i) Lagrange's Eq. of motion for a conservative holonomic dynamical system :-

From eq. (11) we have,

$$\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s}\right\} \delta q_s = 0$$

~~When~~ If the constraints are Holonomic then the constraints are functions of generalized coordinates and time.

Since q_s are independent

$\Rightarrow$ δq_s are independent.

So above eq can be written as,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \quad \text{--- (12)}$$

(11) is $\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1}\right]\delta q_1$

Which is the Lagrange's eq. of motion for a conservative system under holonomic constraints.

ii) Lagrange's Eq. of motion for a conservative System when Constraints

-: are non Holonomic :-

(13) is
$a_{ls}\frac{dq_s}{dt} + a_l = 0$
$a_{ls}\, dq_s + a_l\, dt = 0$
for virtual $a_{ls}\,\delta q_s = 0$

When constraints are non holonomic then eq. of constraints can be written as,

$$a_{ls}\dot{q}_s + a_l = 0 \quad \text{--- (13)}$$

$s = 1, 2, \ldots, n$
$l = 1, 2, \ldots, r \qquad (r < n)$

For virtual variation eq. (13) can be written as,

$$a_{ls}\,\delta q_s = 0 \quad \text{--- (14)}$$

This system has "$n-r$" independent coordinates. For the other δq_s we multiply the first eq. of (14) by λ_1, second eq. by λ_2 and so on and rth eq. by λ_r and then adding all these equations we obtain

$$\lambda_1 a_{1s}\,\delta q_s + \lambda_2 a_{2s}\,\delta q_s + \cdots + \lambda_r a_{rs}\,\delta q_s = 0$$

$$\Rightarrow \lambda_l a_{ls}\,\delta q_s = 0 \quad \text{--- (15)}$$

Subtract eq. (15) from eq. (11) we have

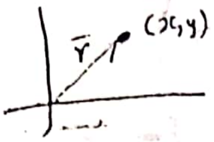
$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} - \lambda_l a_{ls}\right] \delta q_s = 0$$

Since q_s are independent ∴ δq_s are independent. So above eq. becomes,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} - \lambda_l a_{ls} = 0 \quad \text{--- (16)}$$

The eq in (16) is known as Langrange's eq. of motion for a conservative system when constraints are non holonomic.

(iii) Langrange Eq. of motion for a non conservative system when constraints are holonomic:-



$\bar{r} = x\hat{i} + y\hat{j}$

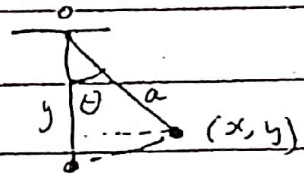
$\bar{v} = \dot{\bar{r}} = \dot{x}\hat{i} + \dot{y}\hat{j}$

$v = \sqrt{\dot{x}^2 + \dot{y}^2}$

$v^2 = \dot{x}^2 + \dot{y}^2$

Applications of Langrange's Eq. of Motion

Q. Derive Langrange's eq. of motion for a simple pendulum.



Let a bob of mass "m" be fasten by a thread of length "a" and is at point (x,y). Let "θ" be the angle of thread with vertical line. Here "θ" is the generalized coordinate. The corresponding Langrang eq. of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{——— (1)}$$

where, $L = T - V$

$L(q_s, \dot{q}_s, t)$ Langrangian must be function of generalized coordinates

as, $K.E = T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

& $P.E = V = -mgy$

From figure,

$x = a\sin\theta, \quad y = a\cos\theta$

$\Rightarrow \dot{x} = a\cos\theta\,\dot{\theta}, \quad \dot{y} = -a\sin\theta\,\dot{\theta}$

$\dot{x}^2 + \dot{y}^2 = a^2\dot{\theta}^2$

Therefore, $T = \frac{1}{2} m a^2 \dot{\theta}^2$

& $V = -mga\cos\theta$

Therefore, $L = \frac{1}{2} m a^2 \dot{\theta}^2 + mga\cos\theta$

$$\Rightarrow \frac{\partial L}{\partial \theta} = -mga\sin\theta$$

and, $\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}$

Put in ① we get,

$\frac{d}{dt}(ma^2\dot{\theta}) + mga\sin\theta = 0$

$\Rightarrow \boxed{ma^2\ddot{\theta} + mga\sin\theta = 0}$

$\Rightarrow \ddot{\theta} = -\frac{g}{a}\sin\theta$

Let, $\frac{g}{a} = K$

$\Rightarrow \ddot{\theta} = -K\sin\theta$

When "θ" is small,

$\sin\theta \approx \theta$

$\Rightarrow \boxed{\ddot{\theta} \propto -\theta}$

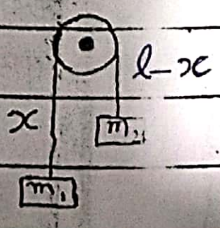
Which is the equation of SHM

Q. Derive Langrage's eq. of motion for atwood's machines

or

Derive Langrage's eq. of motion for the problem where masses passes over a pully.

Sol:



Let the two bodies of masses m_1 and m_2 attached with a string of total length "l". So that

the mass "m_1" is at a distance "x" from the pully. Similarly the mass "m_2" is at a dist. "$l-x$" from the pully. Here "x" is the generalized coordinate. Therefore corresponding Lagranges eq. of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{--- (1)}$$

Where, $L = T - V$

as, T = K.E of system

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2$$

and V = P.E of system

$$= -m_1 g x - m_2 g (l-x)$$

$$\therefore \quad L = \frac{1}{2}(m_1+m_2)\dot{x}^2 + m_1 g x + m_2 g (l-x)$$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} \quad \& \quad \frac{\partial L}{\partial x} = m_1 g - m_2 g$$

So eq. (1) becomes,

$$\frac{d}{dt}\{(m_1+m_2)\dot{x}\} + m_2 g - m_1 g = 0$$

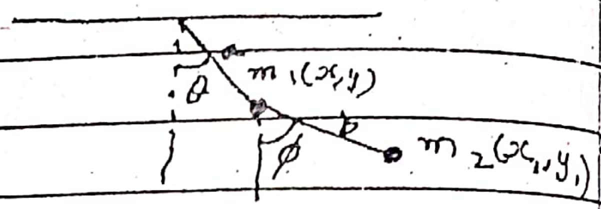
$$\Rightarrow \quad \boxed{\ddot{x} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g}$$

is the required Langrage's eq. of motion.

Q. Find Langrage's eq. of motion for a compound double pendulum.

Sol:-

Let us consider two bobs of masses m_1 and m_2 be fasten by threads with length a & b respectively. Mass m_1 is at point (x, y) making an angle "θ" with vertical. Similarly mass m_2 is at point (x_1, y_1) making an angle "ϕ" with vertical. Further problem is explain in figure. Here θ & ϕ are generalized coordinates. Therefore corresponding Langrage's eq. of motion are.



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{——— (1)}$$

& $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \quad \text{——— (2)}$$

where, $L = T - V$

From figure $x = a\sin\theta$, $y = a\cos\theta$

& $x_1 = x + b\sin\phi$

$x_1 = a\sin\theta + b\sin\phi$

& $y_1 = y + b\cos\phi$

$y_1 = a\cos\theta + b\cos\phi$

$\Rightarrow \dot{x} = a\cos\theta\,\dot{\theta}$, $\dot{y} = -a\sin\theta\,\dot{\theta}$

$\Rightarrow \dot{x}^2 + \dot{y}^2 = a^2\dot{\theta}^2$ —— (*)

& $\dot{x}_1 = a\cos\theta\,\dot{\theta} + b\cos\phi\,\dot{\phi}$

$\dot{y}_1 = -a\sin\theta\,\dot{\theta} - b\sin\phi\,\dot{\phi}$

Squaring & adding

$\dot{x}_1^2 + \dot{y}_1^2 = a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi}\cos(\theta-\phi)$ —(*)

K.E. of the system

$$T = \frac{1}{2}m_1(\dot{x}^2+\dot{y}^2) + \frac{1}{2}m_2(\dot{x}_1^2+\dot{y}_1^2)$$

$$T = \frac{1}{2}m_1(a^2\dot{\theta}^2) + \frac{1}{2}m_2(a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi}\cos(\theta-\phi))$$

& P.E. of the system is,

$$V = -m_1 g y - m_2 g y_1$$

$$V = -m_1 g a\cos\theta - m_2 g(a\cos\theta + b\cos\phi)$$

Therefore,

$$L = T - V$$

$$= \frac{1}{2}m_1(a^2\dot{\theta}^2) + \frac{1}{2}m_2(a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi}\cos(\theta-\phi))$$

$$+ m_1 g a\cos\theta + m_2 g(a\cos\theta + b\cos\phi)$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = m_1 a^2\dot{\theta} + m_2 a^2\dot{\theta} + m_2 ab\dot{\phi}\cos(\theta-\phi)$$

& $$\frac{\partial L}{\partial \theta} = -m_2 ab\dot{\theta}\dot{\phi}\sin(\theta-\phi) - m_1 g a\sin\theta - m_2 g a\sin\theta$$

So eq. (1) becomes,

$$\frac{d}{dt}\left[m_1 a^2\dot{\theta} + m_2 a^2\dot{\theta} + m_2 ab\dot{\phi}\cos(\theta-\phi)\right] + m_2 ab\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

$$+ m_1 g a\sin\theta + m_2 g a\sin\theta = 0$$

$\Rightarrow (m_1+m_2)a^2\ddot{\theta} + m_2 ab\ddot{\phi}\cos(\theta-\phi)$

$- m_2 ab\dot{\phi}\sin(\theta-\phi)(\dot{\theta}-\dot{\phi})$

$+ m_2 ab\dot{\theta}\dot{\phi}\sin(\theta-\phi) + m_1 ga\sin\theta + m_2 ga\sin\theta = 0$

When θ & ϕ are small then,

$\sin\theta \approx \theta$, $\sin(\theta-\phi) = 0$ & $\cos(\theta-\phi) = 1$

Therefore,

$$\boxed{(m_1+m_2)a^2\ddot{\theta} + m_2 ab\ddot{\phi} + m_1 ga\theta + m_2 ga\theta = 0}$$

is the required Langrage's eq. of motion

Now,

$$\frac{\partial L}{\partial \phi} = -ab\dot{\theta}\dot{\phi}\, m_2 \sin(\theta-\phi)(-1) - m_2 gb\sin\phi$$

$$\Rightarrow \frac{\partial L}{\partial \phi} = ab\dot{\theta}\dot{\phi}\, m_2 \sin(\theta-\phi) - m_2 gb\sin\phi$$

and,

$$\frac{\partial L}{\partial \dot{\phi}} = m_2 b^2\dot{\phi} + ab\dot{\theta}\, m_2\cos(\theta-\phi)$$

Now eq (2) becomes,

$$\frac{d}{dt}\left[m_2 b^2\dot{\phi} + ab\dot{\theta}\, m_2\cos(\theta-\phi)\right] - ab\dot{\theta}\dot{\phi}\sin(\theta-\phi)\, m_2 + m_2 gb\sin\phi = 0$$

$$m_2 b^2\ddot{\phi} + ab\ddot{\theta}\cdot m_2\cos(\theta-\phi) - ab\dot{\theta}\, m_2\sin(\theta-\phi)(\dot{\theta}-\dot{\phi}) - ab\, m_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) + m_2 gb\sin\phi = 0$$

When θ and ϕ are small then

$\sin\phi \approx \phi$

, $\sin(\theta-\phi) \approx 0$, $\cos(\theta-\phi) \approx 1$

So above equation reduces to:

$$m_2 b^2 \ddot{\phi} + ab\ddot{\theta} m_2 + m_2 g b \phi = 0$$

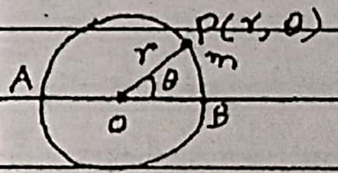
$$\boxed{m_2 b(b\ddot{\phi} + a\ddot{\theta}) + m_2 g b \phi = 0}$$

is the required Lagrange's eq. of motion.

Imp

Q. Find Langrage eq. of motion for a particle moving in a circular orbit.

Sol:



Let a particle of mass "m" is moving in a circular orbit with orbitarly radius "r". Let the particle makes an angle "θ" with ~~a~~ diameter (x-axis) as shown in the figure. Here "r" & "θ" are the generalized coordinates. Therefore corresponding Lagrange's eq.s of motion are,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \text{—(1)}$$

$$\& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{—(2)}$$

Where, $L = T - V$

$T = K.E = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

& $V = mgy$

As we know that,

$x = r\cos\theta$ & $y = r\sin\theta$

$\Rightarrow \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$

$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$

$\Rightarrow \dot{x}^2 + \dot{y}^2 = \dot{r}^2\cos^2\theta + r^2\sin^2\theta\,\dot{\theta}^2 - 2r\dot{r}\sin\theta\cos\theta\,\dot{\theta}$

$+ \dot{r}^2\sin^2\theta + r^2\cos^2\theta\,\dot{\theta}^2 + 2r\dot{r}\sin\theta\cos\theta\,\dot{\theta}$

$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$

Therefore,

$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$

& $V = mgr\sin\theta$

$\therefore L = T - V$

$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\sin\theta$

$\Rightarrow \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ & $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\sin\theta$

& $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ & $\frac{\partial L}{\partial \theta} = -mgr\cos\theta$

(1) $\Rightarrow \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + mg\sin\theta = 0$

$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta = 0$ — (3)

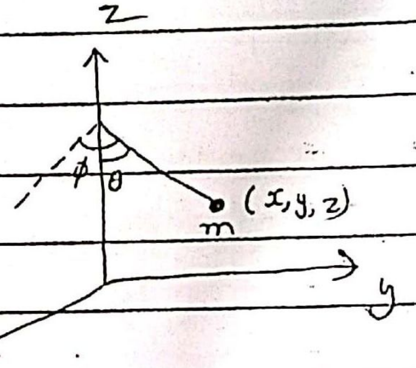
$\frac{d}{dt}(mr^2\dot{\theta}) + mgr\cos\theta = 0$

$\Rightarrow r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\cos\theta = 0$ — (4)

Eq.s ③ & ④ are required Lagrange's eq.s of motion.

Q. Find Lagrange's eq. of motion for a particle moving in sherical equilater or spherical pendulum. Also show that $\theta = \cot^{-1}(A\cos(\phi + B))$

Sol. Let us consider a particle (bob) of mass "m" attached with a rod of length "a" and oscillates under the action of gravity.



Since the particle is constraint to move in a sphere with fixed radius therefore this pendulum is called spherical pendulum. Let at any time "t" the particle is at point P(x, y, z). Therefore,

$x = a\sin\theta\cos\phi \quad , \quad y = a\sin\theta\sin\phi \quad , \quad z = a\cos\theta$

Here θ & ϕ are generalized coordinates.

Therefore corresponding Lagrange's Eq. are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{--- (1)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) - \frac{\partial L}{\partial \phi} = 0 \quad \text{--- (2)}$$

where $L = T - V$

As $T = \frac{1}{2} m(\dot x^2 + \dot y^2 + \dot z^2)$ --- (*)

and $V = -mgz$ --- (*)

Now, $\dot x = a\dot\theta\cos\theta\cos\phi - a\dot\phi\sin\theta\sin\phi$

$\dot y = a\dot\theta\cos\theta\sin\phi + a\dot\phi\sin\theta\cos\phi$

$\dot z = -a\dot\theta\sin\theta$

$\Rightarrow \dot{x}^2+\dot{y}^2+\dot{z}^2 = a^2\dot{\theta}^2\cos^2\theta\cos^2\phi + a^2\dot{\theta}^2\sin^2\theta\sin^2\phi$

$-2a^2\dot{\theta}\dot{\phi}\cos\theta\cos\phi\sin\theta\sin\phi + a^2\dot{\theta}^2\cos^2\theta\sin^2\phi$

$+a^2\dot{\phi}^2\sin^2\theta\cos^2\phi + 2a^2\dot{\theta}\dot{\phi}\cos\phi\cos\theta\sin\phi\sin\theta$

$+a^2\dot{\theta}^2\sin^2\theta$

$\Rightarrow \dot{x}^2+\dot{y}^2+\dot{z}^2 = a^2\dot{\theta}^2 + a^2\dot{\phi}^2\sin^2\theta$

$T = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\dot{\phi}^2\sin^2\theta)$

and $V = -mga\cos\theta$

as, $L = T - V$

$L = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\dot{\phi}^2\sin^2\theta) + mga\cos\theta$

$\frac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta}$; $\frac{\partial L}{\partial\theta} = \frac{1}{2}m(2\dot{\phi}^2\sin\theta\cos\theta) - mga\sin\theta$

and $\frac{\partial L}{\partial\dot{\phi}} = \frac{1}{2}m(2a^2\dot{\phi}\sin^2\theta)$

$= ma^2\dot{\phi}\sin^2\theta$

$\frac{\partial L}{\partial\phi} = 0$

① $\Rightarrow \frac{d}{dt}(ma^2\dot{\theta}) - m\dot{\phi}a^2\sin\theta\cos\theta + mga\sin\theta = 0$

$ma^2\ddot{\theta} - m\dot{\phi}a^2\sin\theta\cos\theta + mga\sin\theta = 0$

$\ddot{\theta} + \frac{g}{a}\sin\theta = \sin\theta\cos\theta\dot{\phi}^2$ —— ③

② $\Rightarrow \frac{d}{dt}[ma^2\dot{\phi}\sin^2\theta] - 0 = 0$

$\Rightarrow \frac{d}{dt}[\dot{\phi}\sin^2\theta] = 0$

Integrate w.r.t. 't'

$\dot{\phi} \sin^2\theta = h$ — (4)

Where 'h' is the constant of integration. Eq.s (3) & (4) represents required eq. of motion.

Next we prove that,

$$\theta = \cot^{-1}[A\cos(\phi + B)]$$

From eq (4)

$$\dot{\phi} = \frac{h}{\sin^2\theta} \quad \text{—— (5)}$$

Put in (3) $\quad \ddot{\theta} + \frac{g}{a}\sin\theta = \sin\theta\cos\theta \frac{h^2}{\sin^4\theta}$

$$\Rightarrow \ddot{\theta} - \frac{h^2\cos\theta}{\sin^3\theta} + \frac{g}{a}\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} - \frac{h^2\cot\theta}{\sin^2\theta} + \frac{g}{a}\sin\theta = 0 \quad \text{—— (6)}$$

Again by Chain rule,

$$\frac{d}{dt} = \frac{d}{d\phi}\frac{d\phi}{dt}$$

$$= \dot{\phi}\frac{d}{d\phi}$$

$$\frac{df}{dt} = \dot{\phi}\frac{df}{d\phi}$$

$$\frac{d}{dt} = \frac{h}{\sin^2\theta}\cdot\frac{d}{d\phi} \quad \text{(7)} \quad \text{using (5)}$$

As $\quad \ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{d}{dt}\left[\frac{d\theta}{dt}\right]$

$$= \frac{d}{dt}\left[\frac{h}{\sin^2\theta}\frac{d\theta}{d\phi}\right] \quad \text{using (7)}$$

$$= \frac{h}{\sin^2\theta}\frac{d}{d\phi}\left[\frac{h}{\sin^2\theta}\frac{d\theta}{d\phi}\right] \quad \text{using (7)}$$

$$= \frac{h^2}{\sin^2\theta}\frac{d}{d\phi}\left(\frac{1}{\sin^2\theta}\frac{d\theta}{d\phi}\right)$$

$$\ddot{\theta} = \frac{-h^2}{\sin^2\theta}\frac{d}{d\phi}\left[\frac{d}{d\phi}(\cot\theta)\right]$$

$$\frac{d}{d\phi}(\cot\theta) = -\operatorname{cosec}^2\theta\frac{d\theta}{d\phi}$$

$$-\frac{d}{d\phi}(\cot\theta) = \frac{1}{\sin^2\theta}\frac{d\theta}{d\phi}$$

$\alpha \pm i\beta$

$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$ (23)

So eq. (B) becomes,

$$\ddot{\theta} - \frac{h^2 \cot\theta}{\sin^2\theta} + \frac{g}{a}\sin\theta = 0$$

$$\frac{-h^2}{\sin^2\theta}\frac{d^2}{d\phi^2}\cot\theta - \frac{h^2\cot\theta}{\sin^2\theta} + \frac{g}{a}\sin\theta = 0$$

$$\Rightarrow \frac{d^2}{d\phi^2}\cot\theta + \cot\theta - \frac{g}{ah^2}\sin^3\theta = 0$$

Since 'θ' is small, therefore neglecting $\sin^3\theta$

$$\frac{d^2}{d\phi^2}\cot\theta + \cot\theta = 0$$

$$\Rightarrow (D^2+1)\cot\theta = 0 \qquad \text{where } D = \frac{d}{d\phi}$$

Chracteristics eq. is,

$$D^2+1 = 0$$

$$\Rightarrow D = \pm i \qquad i = \sqrt{-1}$$

Therefore,

$$\cot\theta = e^{0\phi}\{A_1\cos\phi + A_2\sin\phi\}$$

$$\cot\theta = A_1\cos\phi + A_2\sin\phi$$

Let $A_1 = A\cos B$

$A_2 = -A\sin B$

where $A = \sqrt{A_1^2+A_2^2}$ & $B = \tan^{-1}\left(\frac{-A_2}{A_1}\right)$

Therefore,

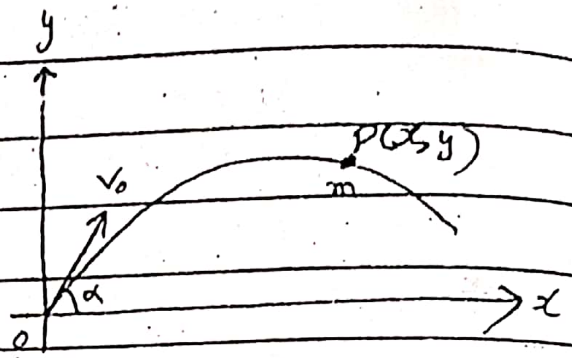
$$\cot\theta = A[\cos\phi\cos B - \sin\phi\sin B]$$

$$= A\cos(\phi+B)$$

$$\Rightarrow \theta = \cot^{-1}[A\cos(\phi+B)]$$

Q. Derive Lagrange eq. of motion for a projectile motion.

Sol:-



Let a particle of mass "m" is projected in space with velocity 'v_0' making an angle "α" with x-axis as shown in figure. Let after time 't' particle is at point (x,y). Here x & y are generalized coordinates. Therefore corresponding Lagrange's eq.s of motion are,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{—①}$$

&

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \quad \text{—②}$$

As, $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ & $V = mgy$

As, $L = T - V$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$\Rightarrow \frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $\frac{\partial L}{\partial x} = 0$

$\frac{\partial L}{\partial \dot{y}} = m\dot{y}$ & $\frac{\partial L}{\partial y} = -mg$

$$① \Rightarrow \frac{d}{dt}(m\dot{x}) = 0$$

$$\Rightarrow \frac{d}{dt}(\dot{x}) = 0$$

$$\Rightarrow \dot{x} = A \quad — ③$$

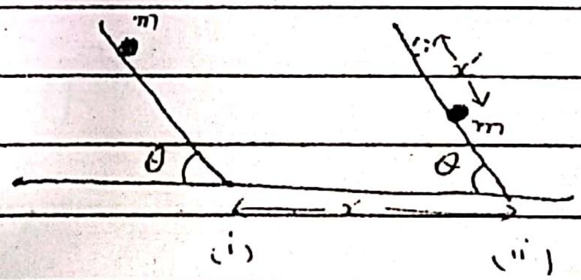
$$② \Rightarrow \frac{d}{dt}(m\dot{y}) + mg = 0$$

$$\Rightarrow \ddot{y} + g = 0 \quad — ④$$

Eq. ③ & ④ are required Langrage eq.s of motions.

Imp
Q: Formulate the problem of a particle sliding down on a smooth inclined plane which is itself moving with a given velocity in a smooth horizontal surface.

Sol:-



Let the x-axis be so choosen that the inclined plane is moving in direction of x-axis. Let "x" be the dist. that the plane has travel in time "t" and "x′" be the distance through which a particle of mass 'm' has slided down in the plane in the same time. Therefore x & x′ are generalized coordinates

of this system. So corresponding Lagrang's eq. of motions are,

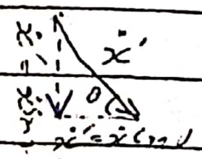
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad -① $$

& $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \frac{\partial L}{\partial x'} = 0 \quad -②$$

From figure (ii) it is clear that actual velocity of particle is given by

$$\vec{V} = \dot{x}\hat{i} + (\dot{x}'\cos\theta\,\hat{i} - \dot{x}'\sin\theta\,\hat{j})$$

$$\vec{V} = (\dot{x} + \dot{x}'\cos\theta)\hat{i} - \dot{x}'\sin\theta\,\hat{j}$$



$$\therefore V^2 = \vec{V}\cdot\vec{V}$$

$$= (\dot{x} + \dot{x}'\cos\theta)^2 + \dot{x}'^2\sin^2\theta$$

$$V^2 = \dot{x}^2 + \dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta \quad —— (*)$$

The K.E & P.E of the system are given by,

$$T = \frac{1}{2}mV^2 + \frac{1}{2}M\dot{x}^2$$

& $V = -mgx'\sin\theta + \text{Constant}$

(Constant term represent the P.E of inclined plane)

Where M is the mass of inclined plane & we have choose the zero level of P.E. at the top of inclined plane.

Therefore,

$$T = \frac{1}{2}m\left[\dot{x}^2 + \dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta\right] + \frac{1}{2}M\dot{x}^2$$

using (*)

As, $L = T - V$

$$= \frac{1}{2} m\left[\dot{x}^2 + \dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta\right]$$

$$+ \frac{1}{2} M\dot{x}^2 + mgx'\sin\theta - \text{Constant}$$

$$\Rightarrow \frac{\partial L}{\partial \dot{x}} = m\dot{x} + m\dot{x}'\cos\theta + M\dot{x}$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}'} = m\dot{x}' + m\dot{x}\cos\theta$$

$$\frac{\partial L}{\partial x'} = mg\sin\theta$$

(1) $\Rightarrow$ $\frac{d}{dt}\left[(m+M)\dot{x} + m\dot{x}'\cos\theta\right] - 0 = 0$

$$(m+M)\dot{x} + m\dot{x}'\cos\theta = h \quad \text{—— (3)}$$

($\because h$ is a const of integration)

(2) $\Rightarrow$

$$\frac{d}{dt}\left[m(\dot{x}' + \dot{x}\cos\theta)\right] - mg\sin\theta = 0$$

$$m(\ddot{x}' + \ddot{x}\cos\theta) - mg\sin\theta = 0$$

$$\Rightarrow \ddot{x}' + \ddot{x}\cos\theta - g\sin\theta = 0 \quad \text{—— (4)}$$

Eq. (3) & (4) are required Lagrange's eq. of motion.

Imp. **:- d – δ Rule :-**

We prove that,

$$d\delta r_i = \delta dr_i$$

Consider a curve r_i,

where, $r_i = r_i(t, \alpha)$ — ①

Where "t" is a parameter described on the curve & "α" names the curve under consideration (α = 0 gives special curve. Diff ① we have

$$dr_i = \frac{\partial r_i}{\partial t}dt + \frac{\partial r_i}{\partial \alpha}d\alpha \quad — \circledast$$

In case α = constant

$$d\alpha = 0.$$

$\therefore \circledast \Rightarrow \quad dr_i = \frac{\partial r_i}{\partial t}dt$ — ②

In case t = constant

$\circledast \Rightarrow \quad \delta r_i = \frac{\partial r_i}{\partial \alpha}\delta\alpha$ — ③

Consider, $d\delta r_i = d[\delta r_i]$

$= d\left(\frac{\partial r_i}{\partial \alpha}\delta\alpha\right)$ using ③

$= \frac{\partial}{\partial t}\left(\frac{\partial r_i}{\partial \alpha}\delta\alpha\right)dt$ using ②

$d\delta r_i = \frac{\partial^2 r_i}{\partial t\,\partial \alpha}\delta\alpha\, dt$ — ④

Now consider,

$$\delta dr_i = \delta(dr_i)$$

$$= \delta\left(\frac{\partial r_i}{\partial t} dt\right) \quad \text{using (2)}$$

$$= \frac{\partial}{\partial \alpha}\left(\frac{\partial r_i}{\partial t} dt\right)\delta\alpha \quad \text{using (3)}$$

$$= \frac{\partial^2 r_i}{\partial \alpha\, \partial t} dt\, \delta\alpha$$

$$\delta dr_i = \frac{\partial^2 r_i}{\partial t\, \partial \alpha} \delta\alpha\, dt \quad \text{--- (5)}$$

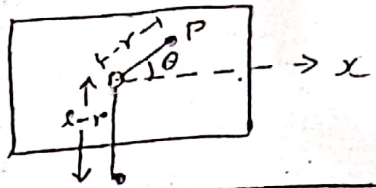
From (4) & (5)

Imp

$$d\delta r_i = \delta dr_i$$

Q. A particle on a smooth horizental table is attached with a string passes through a small hole in the table and carries an equal particle hanging vertically if at any time particle is at a distance "r" from the hole and the string makes an angle "θ" with some fixed horizental line. Find the Lagrange eq. of motion. If the particle on table is initially projected at right angle to the string with velocity $\sqrt{2gh}$ when at a distance "a" from the hole then prove that $\dot{r}^2 = gh\left(1 - a^2/r^2\right) + g(a - r)$

Sol:: Let at any time "t" the particle is at a distance "r" from the hole "o"



We also suppose that length of string is "l". Therefore 2nd particle is at a distance "l-r" from 'O'. We draw a fix line parallel to x-axis and passes through 'O' which makes an angle "θ" with line. OP as shown in figure. Here 'r' & 'θ' are generalized coordinates. Therefore corresponding Lagrange's eq.s of motion are,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \text{---} (1)$$

$$\& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{---} (2)$$

Where, $L = T - V$

For particle "P".

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = 0$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

$$\dot{z} = 0$$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

Therefore, $T_P = K.E \text{ of } P = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

& $V_P = P.E \text{ of } P = 0$ ($\because$ Particle 'P' lies on table)

For particle "Q"

$$x = 0, \quad y = 0, \quad z = l - r$$

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = -\dot{r}$$

$\Rightarrow T_{Q} = \frac{1}{2} m(\dot{r}^2)$

& $V_{Q} = +mg(l-r)$ → Ø

$T = T_P + T_Q$

$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m\dot{r}^2$

$T = \frac{1}{2} m(2\dot{r}^2 + r^2\dot{\theta}^2)$

& $V = V_P + V_Q = 0 + mg(l-r)$

$= +mg(l-r)$

Therefore, $L = T - V$

$$L = \frac{1}{2} m(2\dot{r}^2 + r^2\dot{\theta}^2) - mg(l-r) \quad \text{—} \; (*)$$

$\frac{\partial L}{\partial \dot{r}} = \frac{1}{2} m(4\dot{r}) = 2m\dot{r}$, $\frac{\partial L}{\partial r} = \frac{1}{2} m(2r\dot{\theta}^2) - mg(-1)$

$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg$

$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}(2r^2\dot{\theta}) = mr^2\dot{\theta}$ & $\frac{\partial L}{\partial \theta} = 0$

So eq (1) becomes

$$\frac{d}{dt}(2m\dot{r}) - (mr\dot{\theta}^2 + mg) = 0$$

$2m\ddot{r} - mr\dot{\theta}^2 - mg = 0$

$2\ddot{r} - r\dot{\theta}^2 - g = 0$

$\Rightarrow \ddot{r} = \frac{1}{2}(r\dot{\theta}^2 + g) \quad \text{—} \; (3)$

(2) $\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) - 0 = 0$

Integrating $mr^2\dot{\theta} = \text{Const.} \quad \text{—} \; (4)$

Eq.s (3) & (4) are required Lagrange's eq. of motion

Next we show that,

$$\dot{r}^2 = gh\left(1 - \frac{a^2}{r^2}\right) + g(a-r)$$

Initially given,

$t = 0,\ r = a\ \&\ v = \sqrt{2gh}$

$\Rightarrow r\dot{\theta} = \sqrt{2gh}$

$v = r\omega$
$v = r\frac{d\theta}{dt}$
$v = r\dot{\theta}$

So eq (*) become

$$L = \frac{1}{2}m\left[2(0) + (\sqrt{2gh})^2\right] - mg(l-a)$$

$$L = mgh - mgl + mga \quad \text{— (*)'}$$

From (*) & (*)' we have,

$$\frac{1}{2}m(2\dot{r}^2 + r^2\dot{\theta}^2) - mg(l-r) = mgh - mgl + mga$$

$$\Rightarrow \dot{r}^2 + \tfrac{1}{2}r^2\dot{\theta}^2 = gh - gr + ga \quad \text{— (5)}$$

From eq (4)

$mr^2\dot{\theta} = \text{Constant}$

$mr(r\dot{\theta}) = \text{Constant}$

Initially $ma\sqrt{2gh} = \text{Constant}$ — (6)

From eq (4) & (6) we have,

$ma\sqrt{2gh} = mr^2\dot{\theta}$

$\dot{\theta} = \frac{a}{r^2}\sqrt{2gh}$

Put in (5) $\dot{r}^2 + \frac{1}{2}r^2\left(\frac{a^2}{r^4}2gh\right) = gh - gr + ga$

$$\dot{r}^2 = gh - gr + ga - \frac{a^2}{r^2}gh$$

$$\dot{r}^2 = gh\left(1 - \frac{a^2}{r^2}\right) + g(a-r)$$

Cyclic Coordinates or Ignorable Coordinates:-

Some times it so happen that some of the generalized coordinates say $q_1, q_2, \ldots q_k$ where $k < n$ do not appear in Lagrangian function. Although the generalized velocities corresponding to these coordinates are present in Lagrangian function. Such coordinates are called cylic or ignorable coordinates.

For example if the generalized coordinates of a problem are r, θ with Lagrangian function;

$$L = \frac{1}{2} m (2\dot{r}^2 + r^2\dot{\theta}^2) - mg(l - r) \quad \text{---} \circledast$$

Clearly $\dot{\theta}$ present in "L" but not θ. Therefore "θ" is ignorable coordinate & "r" is non ignorable coordinate.

Routhian Eq. of Motion for Non ignorable Coordinates:-

Let us consider a dynamical system of "N" particles whose configuration at any time "t" is specified by n generalized coordinates $q_1, q_2 \ldots q_n$

Then Lagrange eq. of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \quad \text{--- (1)}$$

Now suppose that we have "k" ignorable coordinates i.e. $q_1, q_2, - q_k$ $(k < n)$, Now after k coordinates all present in "L".

Now $\frac{\partial L}{\partial q_i} = 0$ --- (2)

where, $i = 1, 2, 3, - k$

So corresponding Lagrange eq. of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - 0 = 0$$

$i = 1, 2, - k$

Integrate w.r.t "t".

$$\frac{\partial L}{\partial \dot{q}_i} = p_i \quad \text{--- (3)}$$

Where p_i are constant of integration and called generalized momentum.

The Lagrangian function in this case is,

$$L = L(q_{k+1}, -- q_n, \dot{q}_1, \dot{q}_2, - \dot{q}_n, t)$$

or $L = L(q_j, \dot{q}_s, t)$ where $j = k+1, -- n$

$s = 1, 2, -- n$

Rothcam define a function as,

$$R = L - \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \quad \text{--- (4)}$$

Where, $R = R(q_{k+1}, -- q_n, \dot{q}_{k+1}, - \dot{q}_n, p_1 - p_n, t)$

$z = f(x, y)$

$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$L = L(q_j, \dot{q}_s, t)$

$\delta L = \frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i$

$$R = (q_j, \dot{q}_j, p_i, t) \quad \text{—— (5)}$$

From eq (4)

$$\delta R = \delta L - \delta\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)$$

$$\delta R = \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s - \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i - \dot{q}_i \delta\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

$$= \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j - \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i - \dot{q}_i \delta p_i$$

using (3)

$$\delta R = \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j - \dot{q}_i \delta p_i \quad \text{—— (6)}$$

From eq (5)

$$\delta R = \frac{\partial R}{\partial q_j}\delta q_j + \frac{\partial R}{\partial \dot{q}_j}\delta \dot{q}_j + \frac{\partial R}{\partial p_i}\delta p_i \quad \text{—— (7)}$$

From eq. (6) & (7) we have,

$$\frac{\partial L}{\partial q_j} = \frac{\partial R}{\partial q_j} \quad \text{—— (8)}$$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial R}{\partial \dot{q}_j} \quad \text{—— (9)}$$

$$-\dot{q}_i = \frac{\partial R}{\partial p_i} \quad \text{—— (10)}$$

Eq. (1) for non ignorable coordinates is,

~~$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0$~~

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$j = k+1, \ldots n$

using (8) & (9)

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) - \frac{\partial R}{\partial q_j} = 0 \quad \text{---} (11)$$

$j = k+1, \cdots n$

This eq. is known as Rothean eq. of motion for non ignorable coordinates.

From eq. (10) we have,

$$\dot{q}_i = -\frac{\partial R}{\partial p_i}$$

Integrate w.r.t "t"

$$q_i = \int -\frac{\partial R}{\partial p_i}\, dt \quad \text{---} (12)$$

$i = 1, 2, \cdots k$

Eq. (12) gives value of ignorable coordinates.

Q. For a dynamical system with two degree of freedom the K.E & P.E are given by

$$T = \frac{1}{2}\frac{\dot{q}_1^2}{a+bq_2^2} + \frac{1}{2}\dot{q}_2^2$$

& $V = c + dq_2^2$

where a, b, c & d are constants. Show that eq. of motion of non ignorable coordinate is

$$\ddot{q}_2 + (2d + d\beta^2)\, q_2 = 0$$

where β is generalized momentum.

Also prove that the value of ignorable coordinate is,

$$q_1 = \beta a t + \frac{1}{2}\beta b A^2 t - \frac{\beta b A^2}{4\sqrt{2d+\beta b^2}} \sin 2\left(\sqrt{2d+\beta b^2}\, t + \epsilon\right)$$

Sol:-

As, $L = T - V$

$$= \frac{1}{2}\frac{\dot{q}_1^2}{a+bq_2^2} + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2 \quad \text{——} (1)$$

Clearly q_1 is ignorable & q_2 is non ignorable coordinate

The corresponding Routhean eq. of motion is,

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_2}\right) - \frac{\partial R}{\partial q_2} = 0 \quad \text{——} (1)$$

Where $R = L - \dot{q}_1 \frac{\partial L}{\partial \dot{q}_1}$

From (1)

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\dot{q}_1}{a+bq_2^2}$$

Given $\beta = \frac{\partial L}{\partial \dot{q}_1}$

$$\Rightarrow \beta = \frac{\dot{q}_1}{a+bq_2^2} \qquad \Rightarrow \dot{q}_1 = \beta(a+bq_2^2) \quad \text{——} (*)$$

Therefore,

$$R = \frac{1}{2}\frac{\dot{q}_1^2}{a+bq_2^2} + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2 - \dot{q}_1\beta$$

Using (*) we obtain,

$$R = \frac{\beta^2(a+bq_2^2)^2}{2(a+bq_2^2)} + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2 - \beta^2(a+bq_2^2)$$

$\Rightarrow \frac{\partial R}{\partial \dot{q}_2} = \dot{q}_2$

$\frac{\partial R}{\partial q_2} = b\beta^2 q_2 - 2d q_2 - 2\beta^2 b q_2 = -\beta^2 b q_2 - 2d q_2$

So eq. ① becomes

$\frac{d}{dt}(\dot{q}_2) + \beta^2 b q_2 + 2d q_2 = 0$

$\ddot{q}_2 + (\beta^2 b + 2d) q_2 = 0$ —— ⊛′

Which is required Rothean eq. of motion.

As, $q_i = -\int \frac{\partial R}{\partial p_i} dt$

$q_1 = -\int \frac{\partial R}{\partial \beta} dt$ —— ③ $(\because \beta = p_1)$

Now $\frac{\partial R}{\partial \beta} = -\beta(a + b q_2^2)$

$\therefore$ ③ $\Rightarrow$, $q_1 = \int \beta(a + b q_2^2) dt$ —— ④

From ⊛

$\ddot{q}_2 + (2d + \beta^2 b) q_2 = 0$

$\frac{d^2 q_2}{dt} + (2d + \beta^2 b) q_2 = 0$

$(D^2 + (2d + \beta^2 b)) q_2 = 0 \qquad D = \frac{d}{dt}$

$D^2 + (2d + \beta^2 d) = 0$

$D = \pm \sqrt{2d + \beta^2 b}\, i$

So $q_2 = A_1 \cos\sqrt{2d + \beta^2 b}\, t + A_2 \sin\sqrt{2d + \beta^2 b}\, t$

Put $A_1 = A\sin\xi$ + $A_2 = A\cos\xi$

$$\Rightarrow A = \sqrt{A_1^2 + A_2^2} \quad \& \quad \xi = \tan^{-1}\left(\frac{A_1}{A_2}\right)$$

$$\Rightarrow q_2 = A\left[\sin\xi\cos\sqrt{2d+\beta^2 b}\,t + \cos\xi\sin\sqrt{2d+\beta^2 b}\,t\right]$$

$$q_2 = A\sin\left(\sqrt{2d+\beta^2 b}\,t + \xi\right)$$

$$q_1 = \int \beta\left[a + bA^2\sin^2\left(\sqrt{2d+\beta^2 b}\,t + \xi\right)\right]dt$$

Put in (4)

$$= \int \beta a\,dt + \int \beta b A^2 \sin^2\left(\sqrt{2d+\beta^2 d}\,t + \xi\right)dt$$

$$= \beta a t + \beta b A^2 \int \frac{1 - \cos 2\left(\sqrt{2d+\beta^2 b}\,t + \xi\right)}{2}\,dt$$

$$= \beta a t + \frac{\beta b A^2 t}{2} - \frac{\beta b A^2 \sin 2\left(\sqrt{2d+\beta^2 b}\,t + \xi\right)}{4\sqrt{2d+\beta^2 b}}$$

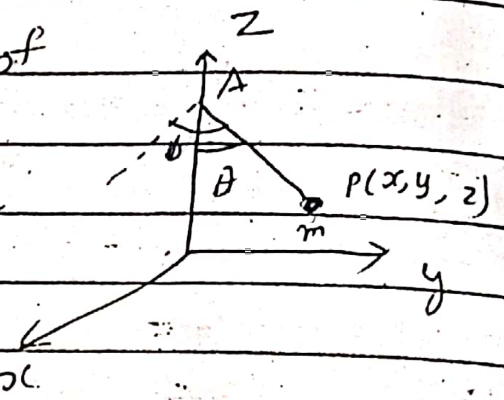
Q. A particle of mass "m" is attached to one end of a rigid rod moving in a three dimensional space under the action of gravity in the form of spherical pendulum. Evaluate Lagrangian function & show that ϕ but not θ is ignorable. Find the Rothean eq. of motion and prove that,

$$\ddot{\theta} - \frac{P_\phi^2\cos\theta}{m^2a^4\sin^3\theta} + \frac{g}{a}\sin\theta = 0$$

where a is the length of rod & P_ϕ is the generalized momentum of ϕ

Sol:-

Let the rod of length "a" is fixed at a point. The particle moves in a spherical pendulum. Further problem is explained in the figure.



Let at any time "t" the particle is at point (x, y, z)

There fore,

$x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$

$\dot{x} = a\cos\theta\cos\phi\,\dot{\theta} - a\sin\theta\sin\phi\,\dot{\phi}$

$\dot{y} = a\sin\theta\cos\phi\,\dot{\phi} + a\cos\theta\sin\phi\,\dot{\theta}$

$\dot{z} = -a\sin\theta\,\dot{\theta}$

& $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = a^2\dot{\theta}^2 + a^2\dot{\phi}^2\sin^2\theta$

$\Rightarrow \quad T = \frac{1}{2} m(a^2\dot{\theta}^2 + a^2\dot{\phi}^2\sin^2\theta)$

& $\quad V = -mgz = -mga\cos\theta$

$\Rightarrow \quad L = T - V$

$L = \frac{1}{2} m(a^2\dot{\theta}^2 + a^2\dot{\phi}^2\sin^2\theta) + mga\cos\theta$

Clearly ϕ but not θ is ignorable. Since 'θ' is non ignorable. Therefore corresponding Rothean eq. of motion is,

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0 \quad \text{——— (1)}$$

Now $\quad \frac{\partial L}{\partial \dot{\phi}} = ma^2\dot{\phi}\sin^2\theta$

Given, $\frac{\partial L}{\partial \dot{\phi}} = p_\phi$

$\Rightarrow p_\phi = m a^2 \dot{\phi} \sin^2\theta$

$\Rightarrow \dot{\phi} = \frac{p_\phi}{m a^2 \sin^2\theta}$ —(2)

As $R = L - \dot{\phi} p_\phi$

$= \frac{1}{2} m(a^2\dot{\theta}^2 + a^2\dot{\phi}^2 \sin^2\theta) + mga\cos\theta - mga\cos\alpha - \dot{\phi} p_\phi$

$= \frac{1}{2} m\left(a^2\dot{\theta}^2 + a^2 \frac{p_\phi^2}{m^2 a^4 \sin^4\theta} \sin^2\theta\right) + mga\cos\theta - \frac{p_\phi^2}{m a^2 \sin^2\theta}$

$= \frac{1}{2} m a^2\dot{\theta}^2 + \frac{p_\phi^2}{2 m a^2 \sin^2\theta} + mga\cos\theta - \frac{p_\phi^2}{m a^2 \sin^2\theta}$

$R = \frac{1}{2} m a^2\dot{\theta}^2 + mga\cos\theta - \frac{1}{2}\frac{p_\phi^2}{m a^2 \sin^2\theta}$

$\Rightarrow \frac{\partial R}{\partial \dot{\theta}} = m a^2 \dot{\theta}$ & $\frac{\partial R}{\partial \theta} = -mga\sin\theta + \frac{p_\phi^2}{2 m a^2}[2\,\mathrm{cosec}^2\theta\cot\theta]$

$\Rightarrow \frac{\partial R}{\partial \theta} = -mga\sin\theta + \frac{p_\phi^2}{m a^2}\frac{\cos\theta}{\sin^3\theta}$

So (1) becomes,

$\frac{d}{dt}(m a^2 \dot{\theta}) + mga\sin\theta - \frac{p_\phi^2 \cos\theta}{m a^2 \sin^2\theta} = 0$

$\Rightarrow \ddot{\theta} + \frac{g}{a}\sin\theta - \frac{p_\phi^2\cos\theta}{m^2 a^4 \sin^3\theta} = 0$

Q. A particle on a smooth horizontal table is attached to a string passes through a small hole in

the table and carry an equal particle hanging vertically. if at any time "t" the particle is at a distance "r" from the hole & string makes an angle 'θ' with some fixed line. Formulate the problem and show that "θ" but not "r" is ignorable. Find the Rothcean eq. of motion for non ignorable coordinates.

GENTLEMAN

M.P.

Hamilton principle / Principle of least action

Imp

Canonical Hamilton's Eq.s of motion:-

For Holonomic Dynamical System:-

Let us consider a dynamical system of "N" particles whose configuration at any time "t" is specified by n generalized coordinates (Lagrangian coordinates) $q_1, q_2 - q_n$. Then Lagrange eq. of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \quad \text{—— (1)}$$

$s = 1, 2, 3, \ldots n$.

Hamilton define a function,

$$H = \dot{q}_s p_s - L \quad \text{—— (2)}$$

$$= H(q_s, p_s, t) \quad \text{—— (3)}$$

$L = L(q_i, \dot{q}_i, t)$

$\Rightarrow \delta L = \frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s + \frac{\partial L}{\partial t}\delta t$

From eq. (2)

$$\delta H = \delta(\dot{q}_s p_s) - \delta L$$

$$= \dot{q}_s \delta p_s + p_s \delta \dot{q}_s - \frac{\partial L}{\partial q_s}\delta q_s - \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s \quad \text{—— (4)}$$

As we know that,

$$\frac{\partial L}{\partial \dot{q}_s} = p_s \quad \text{—— (5)}$$

$\therefore$ (1) $\Rightarrow$ $\frac{d}{dt} p_s - \frac{\partial L}{\partial q_s} = 0$

$$\Rightarrow \frac{\partial L}{\partial q_s} = \dot{p}_s \quad \text{—— (6)}$$

Using (5) & (6) in (4)

$$\delta H = \dot{q}_s \delta p_s + p_s \delta \dot{q}_s - \dot{p}_s \delta q_s - p_s \delta \dot{q}_s$$

$$\delta H = -\dot{p}_s \delta q_s + \dot{q}_s \delta p_s \quad \text{--- (7)}$$

From eq. (3)

$$\delta H = \frac{\partial H}{\partial q_s} \delta q_s + \frac{\partial H}{\partial p_s} \delta p_s \quad \text{--- (8)}$$

From (7) & (8) we obtain,

$$\frac{\partial H}{\partial q_s} = -\dot{p}_s \quad \text{--- (9)}$$

$$\& \quad \frac{\partial H}{\partial p_s} = \dot{q}_s \quad \text{--- (10)}$$

Eq. (9) & (10) are required Hamilton eq.s of motion for a conservative Holonomic dynamical system.

For Non Holonomic Dynamical -: System :-

Let us consider a dynamical system of N particles under non holonomic constraints. Suppose $q_1, q_2, \dots q_n$ are the Lagrangian coordinates of the system then corresponding Lagrange eq.s of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} - \lambda_l a_{ls} = 0 \quad \text{--- (1)}$$

$$s = 1, 2, \dots n$$
$$l = 1, 2, \dots l \quad (l < n)$$

Hamilton define a function,

$$H = \dot{q}_s p_s - L \quad \text{---} \quad (2)$$

$$= H(q_s, p_s, t) \quad \text{---} \quad (3)$$

From (2)

$$\delta H = \delta(\dot{q}_s p_s) - \delta L$$

$$\delta H = \dot{q}_s \delta p_s + p_s \delta \dot{q}_s - \frac{\partial L}{\partial q_s} \delta q_s - \frac{\partial L}{\partial \dot{q}} \delta \dot{q}_s \quad \text{---} \quad (4)$$

as,

$$\frac{\partial L}{\partial \dot{q}_s} = p_s \quad \text{---} \quad (5)$$

Put in (1)

$$\frac{d}{dt}(p_s) - \frac{\partial L}{\partial q_s} - \lambda_e a_{es} = 0$$

$$\Rightarrow \frac{\partial L}{\partial q_s} = \dot{p}_s - \lambda_e a_{es} \quad \text{---} \quad (6)$$

Put (5) & (6) in (4)

$$\delta H = \dot{q}_s \delta p_s + p_s \delta \dot{q}_s - (\dot{p}_s - \lambda_e a_{es}) \delta q_s - p_s \delta \dot{q}_s$$

$$\delta H = -(\dot{p}_s - \lambda_e a_{es}) \delta q_s + \dot{q}_s \delta p_s \quad \text{---} \quad (7)$$

From (3)

$$\delta H = \frac{\partial H}{\partial q_s} \delta q_s + \frac{\partial H}{\partial p_s} \delta p_s \quad \text{---} \quad (8)$$

Equating (7) & (8) we have,

$$\frac{\partial H}{\partial q_s} = -(\dot{p}_s - \lambda_e a_{es}) \quad \text{---} \quad (9)$$

$$\frac{\partial H}{\partial p_s} = \dot{q}_s \quad \text{---} \quad (10)$$

Eq.s (9) & (10) are required Hamilton eq.s of motion for non

Holonomic Constraint.

Q. If a particle of mass "m" is moving in a plane under conservative force with Lagrangian,

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Find the Hamiltonian function and corresponding Hamilton eq. of motion.

Soln

Let p_θ = Momentum of θ

& p_r = Momentum of r

As, $H = \dot{q}_s p_s - L$

So in this case

$$H = \dot{r} p_r + \dot{\theta} p_\theta - \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \quad (*)$$

As, $p_s = \frac{\partial L}{\partial \dot{q}_s}$

$$\Rightarrow p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\Rightarrow \dot{r} = \frac{p_r}{m} \quad (1)$$

and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$$\Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2} \quad (2)$$

$\Rightarrow$ from (*)

$$H = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{1}{2} m \left[\frac{p_r^2}{m^2} + \frac{r^2 p_\theta^2}{m^2 r^4} \right] + V(r)$$

$$H = \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + V(r) \quad \text{—} \; (*)'$$

Since "r" and "θ" are generalized coordinates. Therefore corresponding Hamilton eq.s of motions are,

$$\frac{\partial H}{\partial r} = -\dot{p}_r \quad \text{—} \; (3)$$

$$\frac{\partial H}{\partial p_r} = \dot{r} \quad \text{—} \; (4)$$

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad \text{—} \; (5) \quad \& \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \text{—} \; (6)$$

From $(*)'$

$$\frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + V'(r)$$

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \frac{\partial H}{\partial \theta} = 0, \quad \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

So (3) to (6) becomes,

$$-\frac{p_\theta^2}{mr^3} + V'(r) = -\dot{p}_r \quad \text{—} \; (7)$$

$$\frac{p_r}{m} = \dot{r} \quad \text{—} \; (8) \qquad -\dot{p}_\theta = 0 \Rightarrow \dot{p}_\theta = 0 \quad \text{—} \; (9)$$

$$\frac{p_\theta}{mr^2} = \dot{\theta} \quad \text{—} \; (10)$$

Eq.s (7) to (10) are required Hamilton eq.s of motion.

Q. If,

$\frac{\partial H}{\partial p_s} = \dot{q}_s$ & $\frac{\partial H}{\partial q_s} = -\dot{p}_s$

& $H = p_1 q_1 - p_2 q_2 - a q_1^2 + b q_2^2$

Then prove that,

(i) $\frac{p_2 - b q_2}{q_1} = \text{Constant}$

(ii) $q_1 q_2 = \text{Constant}$ (iii) $\log q_1 - t = \text{Constant}$

Sol:-

Given,

$$\frac{\partial H}{\partial p_s} = \dot{q}_s \Rightarrow \frac{\partial H}{\partial p_1} = \dot{q}_1 \Rightarrow q_1 = \dot{q}_1 \quad \text{--- (1)}$$

$$\therefore \frac{\partial H}{\partial p_2} = \dot{q}_2 \Longrightarrow -q_2 = \dot{q}_2 \quad \text{--- (2)}$$

& $\frac{\partial H}{\partial q_s} = -\dot{p}_s \Rightarrow \frac{\partial H}{\partial q_1} = -\dot{p}_1$

$$p_1 - 2a q_1 = -\dot{p}_1$$

$$\Rightarrow \dot{p}_1 = 2a q_1 - p_1 \quad \text{--- (3)}$$

& $\frac{\partial H}{\partial q_2} = -\dot{p}_2$

$$-p_2 + 2b q_2 = -\dot{p}_2$$

$$\dot{p}_2 = p_2 - 2b q_2 \quad \text{--- (4)}$$

To prove,

(i) $\frac{p_2 - b q_2}{q_1} = \text{Constant}$

Consider, $\frac{d}{dt}\left(\frac{p_2 - b.q_2}{q_1}\right)$

(200 - 25Q)

$$= \frac{q_1 [\dot{p}_2 - b\dot{q}_2] - [p_2 - b q_2] \dot{q}_1}{q_1^2}$$

using ① to ④ in this equatio

$$= \frac{q_1 [p_2 - 2b q_2) + b(q_2)] - [p_2 - b q_2](q_1)}{q_1^2}$$

$$= \frac{q_1 p_2 - 2b q_1 q_2 - b q_1 q_2 - p_2 q_1 + b q_1 q_2}{q_1^2}$$

$$= 0$$

So it is constat

(ii) $\frac{d}{dt}(q_1 q_2) = q_1 \dot{q}_2 + q_2 \dot{q}_1$

$= -q_1 q_2 + q_2 q_1$

$= 0$

So it is $(q_1 q_2)$ is constant.

(iii) $\frac{d}{dt}(\log q_1 - t) = \frac{1}{q_1} \dot{q}_1 - 1$

$= \frac{1}{q_1} q_1 - 1$

$= 1 - 1 = 0$

$\Rightarrow \log q_1 - t$ is constant.

Q. Particle is moving in space with $L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + A\dot{x} + B\dot{y} + C\dot{z} - V$ where A, B & C are the functions of x, y & z. Show that,

$$m\ddot{x} = -\frac{\partial V}{\partial x} + \dot{y}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) + \dot{z}\left(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}\right) \quad -(1)$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} + \dot{z}\left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) + \dot{x}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) \quad -(2)$$

$$m\ddot{z} = -\frac{\partial V}{\partial z} + \dot{x}\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) + \dot{y}\left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right) \quad -(3)$$

Sol:-

Since x, y & z are the generalized coordinates. Therefore corresponding Lagrange's eq. of motion are,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad -(4)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \quad -(5)$$

& $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \quad -(6)$$

Here, $\frac{\partial L}{\partial \dot{x}} = m\dot{x} + A$

& $\frac{\partial L}{\partial x} = \frac{\partial A}{\partial x}\dot{x} + \frac{\partial B}{\partial x}\dot{y} + \frac{\partial C}{\partial x}\dot{z} - \frac{\partial V}{\partial x}$

So (4) becomes,

$$\frac{d}{dt}(m\dot{x} + A) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \quad m\ddot{x} + \dot{A} = \frac{\partial L}{\partial x}$$

$$m\ddot{x} = -\dot{A} + \frac{\partial L}{\partial x}$$

GENTLEMAN

$$m\ddot{x} = -\dot{A} + \frac{\partial A}{\partial x}\dot{x} + \frac{\partial B}{\partial x}\dot{y} + \frac{\partial C}{\partial x}\dot{z} - \frac{\partial V}{\partial x} \quad —(7)$$

As $A = A(x, y, z)$

$$\Rightarrow \dot{A} = \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z}$$

Put in (7)

$$m\ddot{x} = -\frac{\partial A}{\partial x}\dot{x} - \frac{\partial A}{\partial y}\dot{y} - \frac{\partial A}{\partial z}\dot{z} + \frac{\partial A}{\partial x}\dot{x} + \frac{\partial B}{\partial x}\dot{y} + \frac{\partial C}{\partial x}\dot{z} - \frac{\partial V}{x}$$

$$m\ddot{x} = -\frac{\partial V}{\partial x} + \dot{y}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) + \dot{z}\left(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}\right)$$

Similarly by solving eq. (5) & (6) we obtain eq.s (2) & (3).

:Hamilton's Principle:-

The integral $I = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\,dt$, has stationary value along the actual curve C_1 as compare with the neighbouring trajectory having the same values of time at the terminal points.

Proof:-

Let us consider a dynamical system of N particles whose configuration at any time 't' is specified by n generalized coordinates $q_1, q_2, \ldots, q_n$. Suppose that the system is subjected under the ~~constrain~~ conservative holonomic constraints then corresponding

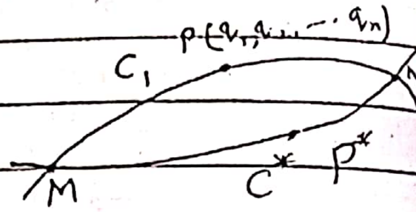
Lagrange's equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \quad \text{—(1)}$$

$s = 1, 2, 3, \ldots n$

Let us consider the actual curve C_1 with terminal points M and N at which the values of time are t_1 and t_2 resp.

Let C^* be the neighbouring trajectory having same values of time at M and N. Let



P be the point on curve C_1 with coordinates $q_1, q_2, - q_n$ and the position of P^* (on C^*) is given by

$q_1 + \delta q_1, \; q_2 + \delta q_2, \; \ldots \; q_n + \delta q_n$

If $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n$ are generalized velocities at P then $\dot{q}_1 + \delta \dot{q}_1, \; \dot{q}_2 + \delta \dot{q}_2, \ldots \; \dot{q}_n + \delta \dot{q}_n$ are generalized velocities at P^*. The value of given integral at P and P^* are

$$I = \int_{t_1}^{t_2} L(q_s, \dot{q}_s, t)\, dt.$$

$$\& \quad I^* = \int_{t_1}^{t_2} L(q_s + \delta q_s, \dot{q}_s + \delta \dot{q}_s, t)\, dt$$

Now, $\delta I = I^* - I$

$$= \int_{t_1}^{t_2} [L(q_s + \delta q_s, \dot{q}_s + \delta \dot{q}_s, t) - L(q_s, \dot{q}_s, t)]\, dt$$

$$\delta I = \int_{t_1}^{t_2} \delta L(q_s, \dot{q}_s, t)\, dt$$

$$\delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s \right] dt$$

$$\delta I = \int_{t_1}^{t_2} \frac{\partial L}{\partial q_s} \delta q_s \, dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s \, dt \quad \text{—(2)}$$

Considere

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s \quad \text{(by d-}\delta\text{ rule)}$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \delta q_s$$

$$(2) \Rightarrow \delta I = \int_{t_1}^{t_2} \frac{\partial L}{\partial q_s} \delta q_s \, dt + \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \delta q_s \right] dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \right] \delta q_s \, dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) dt$$

$$= 0 + \left[\frac{\partial L}{\partial \dot{q}_s} \delta q_s \right]_{t_1}^{t_2} \quad \text{using (1).}$$

$$\delta I = \left(\frac{\partial L}{\partial \dot{q}_s}\right) \delta q_s(t_2) - \frac{\partial L}{\partial \dot{q}_s} \delta q_s(t_1)$$

$$\Rightarrow \delta I = 0$$ (∵ Change in coordinates at terminal points is zero)

Hence the stationary values lies on actual curve are compared with neighbouring trajectory.

Q Derive Hamilton's principle of least action

or

Discuss principle of least action

or

Derive Hamiltons principle using LEOM

Q. Derive Langrage's eq. of motion using Hamilton's Principle.

Sol:-

In Hamilton's Principle,

$$I = \int_{t_1}^{t_2} L(q_s, \dot{q}_s, t)\, dt$$

$$\delta I = \int_{t_1}^{t_2} \delta L(q_s, \dot{q}_s, t)\, dt$$

$$\delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s \right] dt \quad \text{--- (1)}$$

Consider,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s$$

$$\frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \delta q_s$$

Put in (1)

$$\delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \right] \delta q_s\, dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) dt$$

For Hamilton's principle,

$$\delta I = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \right] \delta q_s\, dt + \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s} \delta q_s\right) dt = 0$$

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \right] \delta q_s\, dt + \left[\frac{\partial L}{\partial \dot{q}_s} \delta q_s \right]_{t_1}^{t_2} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \right] \delta q_s\, dt = 0$$

$$\Rightarrow \frac{\partial L}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0$$

Contact Transformation or -: Canonical Transformations :-

Defination:-

Let q_s & p_s where $s = 1, 2, \ldots, n$ be the generalized coordinates and momentum of dynamical system then corresponding Hamilton's eq.s of mation are,

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \quad \text{and} \quad \dot{p}_s = -\frac{\partial H}{\partial q_s}$$

where $H = H(q_s, p_s, t)$

i.e. the Hamiltonian function

The variables satisfing the Hamilton's eq. of motion are called canonically conjugate variables.

$\therefore$ q_s & p_s are canonically conjugate variables. Let $Q_1, Q_2, - Q_n$, $P_1, P_2, \cdots P_n$ be another set of canonically conjugate variables connected by transformations,

$$\left.\begin{aligned} Q_s &= Q_s(q_s, p_s, t) \\ \& \ P_s &= P_s(q_s, p_s, t) \end{aligned}\right\} \quad \text{——} (1) \qquad s = 1, 2, \ldots n$$

Since Q_s and P_s are canonically conjugate variables. Therefore,

$$\frac{\partial K}{\partial P_s} = \dot{Q}_s \quad \& \quad -\frac{\partial H}{\partial Q_s} = \dot{P}_s$$

where K play the role of H

The inverse of transformations defined in (1) exist if

$$J = \frac{\partial (Q_s, P_s)}{\partial (q_s, p_s)} \neq 0$$

By Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

$$\Rightarrow \delta \int_{t_1}^{t_2} (\dot{Q}_s P_s - K) dt = 0 \quad \text{——} (3)$$

Similarly $\delta \int_{t_1}^{t_2} (\dot{q}_s p_s - H) dt = 0$ —— (2)

We observe that,

(i) In both cases the variation at the end points is zero.

(ii) The simultaneous existance of (2) & (3) does not means that both ~~results~~ integrals are equal but they ~~inte~~ differ at the most by total time derivative of an orbitrary function F. This F is called generalized function.

A transformation defined in (1) is canonical or contact if

$p\delta q - P\delta Q = 0$

or $pdq - PdQ = 0$

is an exact differential equation.

Imp

Q. Show that transformation

$Q = \frac{1}{2}(p^2 + q^2)$ & $P = -\tan^{-1}(q/p)$

is canonical.

Sol:-

$dQ = \frac{1}{2}[2p\,dp + 2q\,dq] = p\,dp + q\,dq$

As we know that a transformation is canonical if,

$pdq - PdQ = 0$

is an exact diff. equation.

Consider,

$pdq - PdQ = pdq + \tan^{-1}(q/p)(p\,dp + q\,dq)$

$= [p + q\tan^{-1}(q/p)]\,dq + [p\tan^{-1}(q/p)]\,dp$

Let $M = p + q\tan^{-1}(q/p)$ & $N = p\tan^{-1}(q/p)$

$$\frac{\partial M}{\partial p} = 1 + q\,\frac{1}{1+q^2/p^2}\left(\frac{-q}{p^2}\right) = 1 + \frac{q^2}{p^2+q^2} = \frac{p^2}{p^2+q^2}$$

$$\frac{\partial N}{\partial q} = p\,\frac{1}{1+\frac{q^2}{p^2}}\,\frac{1}{p} = \frac{p^2}{p^2+q^2}$$

as, $\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$

$\Rightarrow$ $pdq - PdQ = 0$ is an exact diff. eq.

So therefore given transformation is canonical.

Generating Functions :-

$q\ p \quad Q\ P$

There are four types of generating functions.

(i) $F_1(q_s, Q_s, t)$

(ii) $F_2(q_s, P_s, t)$

(iii) $F_3(p_s, Q_s, t)$

(iv) $F_4(p_s, P_s, t)$

Case - I

$$F = F_1(q_s, Q_s, t)$$

$$\Rightarrow dF = dF_1 \quad \text{--- (1)}$$

As we know that,

$$dF = p_s\, dq_s - P_s\, dQ_s - (H-K)\, dt \quad \text{--- (2)}$$

& $$dF_1 = \frac{\partial F_1}{\partial q_s} dq_s + \frac{\partial F_1}{\partial Q_s} dQ_s + \frac{\partial F_1}{\partial t} dt \quad \text{--- (3)}$$

From (1), (2) & (3) we have,

$$p_s\, dq_s - P_s\, dQ_s - (H-K)\, dt = \frac{\partial F_1}{\partial q_s} dq_s + \frac{\partial F_1}{\partial Q_s} dQ_s + \frac{\partial F_1}{\partial t} dt$$

Equating the differential coefficients

$$\frac{\partial F_1}{\partial q_s} = p_s \quad \text{--- (a)}$$

$$\frac{\partial F_1}{\partial Q_s} = -P_s \quad \text{--- (b)}$$

& $$\frac{\partial F_1}{\partial t} = K - H \quad \text{--- (c)}$$

If F_1 is independent of t explicitly then,

$$K = H$$

Case – II

$$F = F_2(q_s, P_s, t)$$

$$\Rightarrow \quad dF = dF_2 \quad \text{—} (1)$$

As we know that,

$$dF = p_s dq_s - P_s dQ_s - (H-K)dt \quad \text{—} (2)$$

and $$dF_2 = \frac{\partial F_2}{\partial q_s} dq_s + \frac{\partial F_2}{\partial P_s} dP_s + \frac{\partial F_2}{\partial t} dt \quad \text{—} (3)$$

We want to cancel out the terms $P_s dQ_s$ therefore we put,

$$F_2 = F + P_s Q_s$$

$$\Rightarrow \quad dF_2 = dF + P_s dQ_s + Q_s dP_s$$

using (2) & (3)

$$\frac{\partial F_2}{\partial q_s} dq_s + \frac{\partial F_2}{\partial P_s} dP_s + \frac{\partial F_2}{\partial t} dt = p_s dq_s - P_s dQ_s - (H-K)dt + P_s dQ_s + Q_s dP_s$$

Equating differential coefficients,

$$\frac{\partial F_2}{\partial q_s} = p_s \quad \text{——} (a)$$

$$\frac{\partial F_2}{\partial P_s} = Q_s \quad \text{——} (b)$$

$$\frac{\partial F_2}{\partial t} = K - H \quad \text{——} (c)$$

If F_2 is independent of "t" explicitly then, $K = H$

Case – III

$$F = F_3(p_s, Q_s, t)$$

$$\Rightarrow dF = dF_3$$

As we know that,

(Malik ~ 1)

$dF = p_s dq_s - P_s dQ_s - (H-K)dt$ —— ②

As, $dF_3 = \frac{\partial F_3}{\partial p_s} dp_s + \frac{\partial F_3}{\partial Q_s} dQ_s + \frac{\partial F_3}{\partial t} dt$

We want to cancel out the p term $p_s dq_s$

Put, $F_3 = F - p_s q_s$

$dF_3 = dF - p_s dq_s - q_s dp_s$

using ② & ③

$\frac{\partial F_3}{\partial p_s} dp_s + \frac{\partial F_3}{\partial Q_s} dQ_s + \frac{\partial F_3}{\partial t} dt = p_s dq_s - P_s dQ_s - (H-K)dt - p_s dq_s - q_s dp_s$

Equating differential coefficients, we have;

$\frac{\partial F_3}{\partial p_s} = -q_s$ —— (a)

$\frac{\partial F_3}{\partial Q_s} = -P_s$ —— (b)

$\frac{\partial F}{\partial t} = K - H$ —— (c)

If F_3 is independent of 't' explicitly then $K = H$

Case - IV

$F = F_4(p_s, P_s, t)$

$dF = dF_4$ —— ①

As we know that,

$dF = p_s dq_s - P_s dQ_s - (H-K)dt$ —— ②

As,

$dF_4 = \frac{\partial F_4}{\partial p_s} dp_s + \frac{\partial F_4}{\partial P_s} dP_s + \frac{\partial F_4}{\partial t} dt$ —— ③

We want to cancel out the terms $p_s dq_s$ & $P_s dQ_s$.

Therefore we put,

$$F_4 = F - p_s q_s + P_s Q_s$$

$$dF_4 = dF - p_s dq_s - q_s dp_s + P_s dQ_s + Q_s dP_s$$

Using ② & ③

$$\frac{\partial F_4}{\partial p_s} dp_s + \frac{\partial F_4}{\partial P_s} dP_s + \frac{\partial F_4}{\partial t} dt = p_s dq_s - P_s dQ_s - (H-K)dt - p_s dq_s - q_s dp_s + P_s dQ_s + Q_s dP_s$$

Equating coefficients of differentials

$$\frac{\partial F_4}{\partial p_s} = -q_s \quad \text{--- (a)}$$

$$\frac{\partial F_4}{\partial P_s} = Q_s \quad \text{--- (b)}$$

$$\frac{\partial F_4}{\partial t} = K - H \quad \text{--- (c)}$$

If F_4 is independent of t explicitly then

$$K = H$$

Q. Show that the transformation

$$Q = \cot^{-1}[p/m\omega q] \quad \text{--- ①}$$

$$P = \frac{p^2 + \omega^2 m^2 q^2}{2m\omega} \quad \text{--- ②}$$

is canonical also find generating functions.

Sol:-

As we know that a transformation is canonical if,

$p\,dq - P\,dQ = 0$ is an exact diff. eq.

Now,

$$dQ = \frac{-1}{1+\left(\frac{p}{m\omega q}\right)^2} \cdot \frac{1}{m\omega}\left(\frac{q\,dp - p\,dq}{q^2}\right)$$

$$= \frac{1}{\frac{m^2\omega^2q^2+p^2}{m^2\omega^2q^2}} \cdot \frac{1}{m\omega q^2}(q\,dp - p\,dq)$$

$$dQ = \frac{-m\omega}{p^2+m^2\omega^2q^2}(q\,dp - p\,dq)$$

Consider,

$$p\,dq - P\,dQ = 0$$

$$p\,dq + \frac{p^2+m^2\omega^2q^2}{2m\omega} \cdot \frac{m\omega}{p^2+m^2\omega^2q^2}(q\,dp - p\,dq) = 0$$

$$p\,dq + \frac{q}{2}\,dp - \frac{p}{2}\,dq = 0$$

$$\left(p - \frac{p}{2}\right)dq + \frac{q}{2}\,dp = 0$$

$$\frac{p}{2}\,dq + \frac{q}{2}\,dp = 0$$

$$\Rightarrow \quad p\,dq + q\,dp = 0$$

Let $M = p$ & $N = q$

$$\frac{\partial M}{\partial p} = 1 \qquad \frac{\partial N}{\partial q} = 1$$

$$\Rightarrow \quad \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$$

So given transformation is canonical.

Let, $I_1 = \int p\,dq = p\int dq = pq$

$I_2 = \int q\,dp = q\int dp = qp = pq$

$\therefore \quad F = pq \qquad (F = I_1 = I_2)$

$F_1 = F_1(q, Q, t)$

As, $F_1 = F$

$\Rightarrow F_1 = pq$

From (I)

$$\cot Q = \frac{p}{m\omega q} \Rightarrow p = m\omega q \cot Q$$

$\therefore \quad F_1 = m\omega q^2 \cot Q$

$F_2 = F_2(q, P, t)$

As: $F_2 = F + PQ$

$F_2 = pq + PQ$

$$F_2 = pq + P\cot^{-1}\left(\frac{p}{m\omega q}\right)$$

From (2) $p^2 = 2m\omega P - m^2\omega^2 q^2$

$\Rightarrow p = \sqrt{2m\omega P - m^2\omega^2 q^2}$

$$F_2 = q\sqrt{2m\omega P - m^2\omega^2 q^2} + P\cot^{-1}\left[\frac{\sqrt{2m\omega P - m^2\omega^2 q^2}}{m\omega q}\right]$$

Case - III $\quad F_3 = F_3(p, Q, t)$

as $F_3 = F - pq$

$F_3 = pq - pq = 0$

Case - IV. $\quad F_4 = F_4(p, P, t)$

& as $F_4 = F + PQ - pq$

$F_4 = pq + PQ - pq$

$\Rightarrow \quad F_4 = PQ$

$$F_4 = P\cot^{-1}\left(\frac{p}{m\omega q}\right)$$

From (2) $\quad m^2\omega^2 q^2 = 2m\omega P - p^2$

$$q = \frac{\sqrt{2m\omega P - p^2}}{m\omega}$$

$$F_4 = P\cot^{-1}\left[\frac{p}{m\omega\,\frac{\sqrt{2m\omega P - p^2}}{m\omega}}\right]$$

$$= P\cot^{-1}\left[\frac{p}{\sqrt{2m\omega P - p^2}}\right]$$

Q. Write Hamilton's eq. of motion using generating function F_2 of the transformation defined in previous question.

Sol:-

First we show that transformation is canonical & find F_2

$$F_2 = q\sqrt{2m\omega P - m^2\omega^2 q^2} + P\cot^{-1}\left[\frac{\sqrt{2m\omega P - m^2\omega^2 q^2}}{m\omega q}\right]$$

the corresponding Hamilton's eqs of motion are,

$$\frac{\partial F_2}{\partial q} = p \quad \text{--- (1)} \qquad \& \qquad \frac{\partial F_2}{\partial P} = Q \quad \text{--- (2)}$$

(80)

Q. Show that the transformation

$Q = \log(1 + \sqrt{q}\cos p)$ — (1)

& $P = 2\sqrt{q}\sin p(1 + \sqrt{q}\cos p)$ — (2)

is canonical. Also find F_3.

Sol:-

As we know that transformation is canonical if

$pdq - PdQ = 0$ is an exact diff eq.

Now,

$$dQ = \frac{1}{1+\sqrt{q}\cos p}\left\{0 - \sqrt{q}\sin p\,dp + \frac{1}{2\sqrt{q}}\cos p\,dq\right\}$$

$$dQ = \frac{1}{1+\sqrt{q}\cos p}\left[\frac{-2q\sin p\,dp + \cos p\,dq}{2\sqrt{q}}\right]$$

$$dQ = \frac{\cos p\,dq - 2q\sin p\,dp}{2\sqrt{q}(1+\sqrt{q}\cos p)}$$

Consider,

$$pdq - 2\sqrt{q}\sin p(1+\sqrt{q}\cos p)\frac{\cos p\,dq - 2q\sin p\,dp}{2\sqrt{q}(1+\sqrt{q}\cos p)}$$

$$pdq - \sin p(\cos p\,dq - 2q\sin p\,dp) = 0$$

$$pdq - \sin p\cos p\,dq + 2q\sin^2 p\,dp = 0$$

$$(p - \sin p\cos p)\,dq + 2q\sin^2 p\,dp = 0$$

$M = p - \sin p\cos p$ $\qquad\qquad$ $N = 2q\sin^2 p$

$$\frac{\partial M}{\partial p} = 1 - (\cos p\cos p - \sin p\sin p)$$

$$= 1 - (\cos^2 p - \sin^2 p) = \sin^2 p + \sin^2 p$$

$$= 2\sin^2 p$$

$$\frac{\partial N}{\partial q} = 2\sin^2 p$$

$$\Rightarrow \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$$

So given transformation is canonical.

Let,

$$I_1 = \int (p - \sin p \cos p)\, dq$$

$$= p - \sin p \cos p \int dq$$

$$I_1 = pq - q\sin p \cos p.$$

$$I_2 = \int 2q \sin^2 p \, dp = 2q \int \frac{1 - \cos 2p}{2} \, dp$$

$$= q\left(p - \frac{\sin^2 p}{2}\right)$$

$$= q\left[p - \frac{2\sin p \cos p}{2}\right]$$

$$I_2 = pq - q\sin p \cos p$$

$$F = I_1 = I_2 = pq - q\sin p \cos p$$

$$F = pq - q/_2 \sin 2p$$

$$F_3 = F_3(p, Q, t)$$

As $F_3 = F - pq$

$$F_3 = pq - \frac{q}{2}\sin 2p - pq$$

p#231

$$F_3 = -\frac{q}{2}\sin 2p$$

From (1)

$$e^Q = 1 + \sqrt{q}\cos p$$

$$\sqrt{q} = \frac{e^Q - 1}{\cos p}$$

$$q = \left(\frac{e^Q - 1}{\cos p}\right)^2$$

$$F_3 = -\frac{1}{2}\left(\frac{e^Q - 1}{\cos p}\right)^2 2\sin p\cos p$$

$$F_3 = -\tan p\,(e^Q - 1)^2$$

-: Langrange Bracket :-

If q_s & p_s are functions of u & v

i.e. $q_s = q_s(u, v)$

& $p_s = p_s(u, v)$

then the expression,

$$\frac{\partial q_s}{\partial u}\frac{\partial p_s}{\partial v} - \frac{\partial q_s}{\partial v}\frac{\partial p_s}{\partial u}$$

is called Lagrange bracket of u & v

It is usually denoted by, $\{u, v\}$ or $\{u, v\}_{q,p}$

-: Properties of Lagrange's Bracket :-

(i) $\{u, v\} = -\{v, u\}$

$$L.H.S = \{u, v\} = \frac{\partial q_s}{\partial u}\frac{\partial p_s}{\partial v} - \frac{\partial q_s}{\partial v}\frac{\partial p_s}{\partial u}$$

{These Notes are Lectures Delivered by Sir Saqib.}

$$\frac{\partial q_s}{\partial q_i} = \begin{cases} 1 & \text{if } s = i \\ 0 & \text{if } s \neq i \end{cases} = \delta_{si}$$

$$= -\left(\frac{\partial q_s}{\partial v} \frac{\partial p_s}{\partial u} - \frac{\partial q_s}{\partial u} \frac{\partial p_s}{\partial v} \right)$$

$$= -\{v, u\}$$

$$\therefore \quad \{u, v\} = -\{v, u\}$$

(ii) $\{q_i, q_j\} = 0$

As, $\{u, v\} = \frac{\partial q_s}{\partial u} \frac{\partial p_s}{\partial v} - \frac{\partial q_s}{\partial v} \frac{\partial p_s}{\partial u}$

$$\Rightarrow \{q_i, q_j\} = \frac{\partial q_s}{\partial q_i} \frac{\partial p_s}{\partial q_j} - \frac{\partial q_s}{\partial q_j} \frac{\partial p_s}{\partial q_i}$$

$$= \delta_{si}(0) - \delta_{sj}(0)$$

$$\{q_i, q_j\} = 0$$

(iii) $\{q_i, p_j\} = \delta_{ij}$

As, $\{u, v\} = \frac{\partial q_s}{\partial u} \frac{\partial p_s}{\partial v} - \frac{\partial q_s}{\partial v} \frac{\partial p_s}{\partial u}$

$$\Rightarrow \{q_i, p_j\} = \frac{\partial q_s}{\partial q_i} \frac{\partial p_s}{\partial p_j} - \frac{\partial q_s}{\partial p_i} \frac{\partial p_s}{\partial q_i}$$

$$= \delta_{si} \delta_{sj} - (0)(0)$$

$$= \delta_{ij} - 0$$

$$= \delta_{ij}$$

Also $\{p_i, q_j\} = -\delta_{ij}$

Q. Define bilinear covariant and show that the expression,

$$\sum_{i=1}^{n} (\delta p_i dq_i - \delta q_i dp_i)$$

is invariant under a canonical transformation.

Sol:-

The relation,

$$\sum_{i=1}^{n} (\delta p_i dq_i - \delta q_i dp_i) \quad \text{—— (1)}$$

is linear in δp_i and dp_i. Also it is linear in δq_i & dq_i. Therefore it is called bilinear covariant.

Next we want to show relation (1) is invariant under a canonical transformation.

Consider the transformation

$$p_i = p_i (Q_k, P_k)$$

$$q_i = q_i (Q_k, P_k)$$

then,

$$dp_i = \frac{\partial p_i}{\partial Q_k} dQ_k + \frac{\partial p_i}{\partial P_k} dP_k$$

&

$$dq_i = \frac{\partial q_i}{\partial Q_k} dQ_k + \frac{\partial q_i}{\partial P_k} dP_k$$

$$(1) \Rightarrow \sum (\delta p_i dq_i - \delta q_i dp_i) = \sum$$

$$= \sum \left[\left(\frac{\partial p_i}{\partial Q_k} \delta Q_k + \frac{\partial p_i}{\partial P_k} \delta P_k \right) \left(\frac{\partial q_i}{\partial Q_m} dQ_m + \frac{\partial q_i}{\partial P_m} dP_m \right) \right.$$

$$- \left(\frac{\partial q_i}{\partial Q_k} \delta Q_k + \frac{\partial q_i}{\partial P_k} \delta P_k \right) \left(\frac{\partial p_i}{\partial Q_m} dQ_m + \frac{\partial p_i}{\partial P_m} dP_m \right)$$

$$= \sum \Big[\Big(\frac{\partial q_i}{\partial Q_m} \frac{\partial p_i}{\partial Q_k} \delta Q_k dQ_m + \frac{\partial q_i}{\partial P_m} \frac{\partial p_i}{\partial Q_k} \delta Q_k dP_m$$

$$+ \frac{\partial q_i}{\partial Q_m} \frac{\partial p_i}{\partial P_k} \delta P_k dQ_m + \frac{\partial q_i}{\partial P_m} \frac{\partial p_i}{\partial P_k} \delta P_k dP_m \Big)$$

$$- \Big(\frac{\partial q_i}{\partial Q_k} \frac{\partial p_i}{\partial Q_m} \delta Q_k dQ_m + \frac{\partial q_i}{\partial Q_k} \frac{\partial p_i}{\partial P_m} \delta Q_k dP_m$$

$$+ \frac{\partial q_i}{\partial P_k} \frac{\partial P_i}{\partial Q_m} \delta P_k dQ_m + \frac{\partial q_i}{\partial P_k} \frac{\partial p_i}{\partial P_m} \delta P_k \delta P_m \Big) \Big]$$

$$= \sum \Big[\Big(\frac{\partial q_i}{\partial Q_m} \frac{\partial p_i}{\partial Q_k} - \frac{\partial q_i}{\partial Q_k} \frac{\partial p_i}{\partial Q_m} \Big) \delta Q_k dQ_m$$

$$+ \Big(\frac{\partial q_i}{\partial P_m} \frac{\partial p_i}{\partial Q_k} - \frac{\partial q_i}{\partial Q_k} \frac{\partial p_i}{\partial P_m} \Big) \delta Q_k dP_m$$

$$+ \Big(\frac{\partial q_i}{\partial Q_m} \frac{\partial p_i}{\partial P_k} - \frac{\partial q_i}{\partial P_k} \frac{\partial p_i}{\partial P_m} \Big) \delta P_k dP_m \Big]$$

$$= \sum \Big[\{Q_m, Q_k\} \delta Q_k dQ_m + \{P_m, Q_k\} \delta Q_k dP_m$$

$$+ \{Q_m, P_k\} \delta P_k dQ_m + \{P_m, P_k\} \delta P_k dP_m \Big]$$

$$= \sum \Big[-\delta_{mk} \delta Q_k dP_m + \delta_{mk} \delta P_k dQ_m \Big]$$

Put $m = k$

$$= \sum \big[\delta P_k dQ_k - \delta Q_k dP_k \big]$$

$$\Rightarrow \sum (\delta p_i dq_i - \delta q_i dp_i) = \sum (\delta P_k dQ_k - \delta Q_k dP_k)$$

which shows that relation ① is invariant under a canonical transformation.

GENTLEMAN

:- Poisson Bracket :-

Suppose $u = u(q_i, p_i)$ & $v = v(q_i, p_i)$ $i = 1, 2, \ldots n$ then Poisson bracket of u and v is denoted by $[u, v]$ and is defined as,

$$[u, v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

Properties :-

(i) $[u, v] = -[v, u]$.

As, $$[u, v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

$$= -\left[\frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i} - \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}\right]$$

$$= -[v, u]$$

$$\Rightarrow [u, v] = -[v, u]$$

(ii). $[u, u] = 0 = [v, v]$

As, $$[u, v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

Now, $$[u, u] = \frac{\partial u}{\partial q_i}\frac{\partial u}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial u}{\partial q_i}$$

$$= 0$$

Similarly $[v, v] = 0$

Imp

(iii) $[u, v+w] = [u,v] + [u,w]$

Consider,

$$[u, v+w] = \frac{\partial u}{\partial q_i}\frac{\partial}{\partial p_i}(v+w) - \frac{\partial u}{\partial p_i}\frac{\partial}{\partial q_i}(v+w)$$

$$= \frac{\partial u}{\partial q_i}\left(\frac{\partial v}{\partial p_i} + \frac{\partial w}{\partial p_i}\right) - \frac{\partial u}{\partial p_i}\left(\frac{\partial v}{\partial q_i} + \frac{\partial w}{\partial q_i}\right)$$

$$= \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} + \frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}$$

$$\Rightarrow [u, v+w] = [u,v] + [u,w]$$

Imp

(iv) $\frac{\partial}{\partial t}[u,v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right]$

Consider, $[u,v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$

$$\Rightarrow \frac{\partial}{\partial t}[u,v] = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}\right) - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right)$$

$$= \frac{\partial^2 u}{\partial t \partial q_i}\cdot\frac{\partial v}{\partial p_i} + \frac{\partial u}{\partial q_i}\frac{\partial^2 v}{\partial t \partial p_i} - \frac{\partial^2 u}{\partial t \partial p_i}\frac{\partial v}{\partial q_i}$$

$$- \frac{\partial u}{\partial p_i}\frac{\partial^2 v}{\partial t \partial v_i}$$

$$\frac{\partial}{\partial t}[u,v] = \frac{\partial}{\partial q_i}\left(\frac{\partial u}{\partial t}\right)\frac{\partial v}{\partial p_i} - \frac{\partial}{\partial p_i}\left(\frac{\partial u}{\partial t}\right)\frac{\partial v}{\partial q_i}$$

$$+ \frac{\partial u}{\partial q_i}\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial t}\right) - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\left(\frac{\partial v}{\partial t}\right)$$

$$\Rightarrow \frac{\partial}{\partial t}[u,v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right]$$

(v) $[q_i, q_j] = 0$

Consider;

$$[q_i, q_j] = \frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k}$$

$$= \delta_{ik}(0) - (0)\,\delta_{jk}$$

$$[q_i, q_j] = 0$$

Similarly, $[p_i, p_j] = 0$

(vi) $[q_i, p_j] = \delta_{ij}$

$$[q_i, p_j] = \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}$$

$$= \delta_{ik}\,\delta_{jk} - 0$$

$$= \delta_{ij}$$

Q. Prove that Poisson brackets are invariant under canonical transformation.

Sol:-

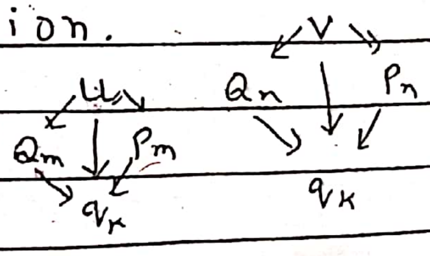
Let,

$$\left.\begin{aligned} Q_m &= Q_m(q_k, p_k) \\ P_m &= P_m(q_k, p_k) \end{aligned}\right\} \quad \text{—— (1)}$$

be a canonical transformation.

We have to prove that,

$$[u, v]_{q,p} = [u, v]_{Q,P}$$



Consider,

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial v}{\partial q_k}$$

$$= \left(\frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial q_k} + \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial q_k}\right)\left(\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial p_k} + \frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial p_k}\right)$$

$$- \left(\frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial p_k} + \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial p_k}\right)\left(\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial q_k} + \frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial q_k}\right)$$

$$= \frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial q_k}\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial p_k} + \frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial q_k}\frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial p_k}$$

$$+ \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial q_k}\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial p_k} + \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial q_k}\frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial p_k}$$

$$- \frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial p_k}\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial q_k} - \frac{\partial u}{\partial Q_m}\frac{\partial Q_m}{\partial p_k}\frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial q_k}$$

$$- \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial p_k}\frac{\partial v}{\partial Q_n}\frac{\partial Q_n}{\partial q_k} - \frac{\partial u}{\partial P_m}\frac{\partial P_m}{\partial p_k}\frac{\partial v}{\partial P_n}\frac{\partial P_n}{\partial q_k}$$

$$= \frac{\partial u}{\partial Q_m}\frac{\partial v}{\partial Q_n}\left(\frac{\partial Q_m}{\partial q_k}\frac{\partial Q_n}{\partial p_k} - \frac{\partial Q_m}{\partial p_k}\frac{\partial Q_n}{\partial q_k}\right)$$

$$+ \frac{\partial u}{\partial Q_m}\frac{\partial v}{\partial P_n}\left(\frac{\partial Q_m}{\partial q_k}\frac{\partial P_n}{\partial p_k} - \frac{\partial P_m}{\partial p_k}\frac{\partial P_n}{\partial q_k}\right)$$

$$+ \frac{\partial u}{\partial P_m}\frac{\partial v}{\partial Q_n}\left(\frac{\partial P_m}{\partial q_k}\frac{\partial Q_n}{\partial p_k} - \frac{\partial P_m}{\partial p_k}\frac{\partial Q_n}{\partial q_k}\right)$$

$$+ \frac{\partial u}{\partial P_m}\frac{\partial v}{\partial P_n}\left(\frac{\partial P_m}{\partial q_k}\frac{\partial P_n}{\partial p_k} - \frac{\partial P_n}{\partial p_k}\frac{\partial P_n}{\partial q_k}\right)$$

$$= \frac{\partial u}{\partial Q_m}\frac{\partial v}{\partial Q_n}[Q_m, Q_n] + \frac{\partial u}{\partial Q_m}\frac{\partial v}{\partial P_n}[Q_m, P_n]$$

$$+ \frac{\partial u}{\partial P_m}\frac{\partial v}{\partial Q_n}[P_m, Q_n] + \frac{\partial u}{\partial P_m}\frac{\partial v}{\partial P_n}[P_m, P_n]$$

$$= \frac{\partial u}{\partial Q_m}\frac{\partial v}{\partial P_n}\delta_{mn} - \frac{\partial u}{\partial P_m}\frac{\partial v}{\partial Q_n}\delta_{mn}$$

Put $m = n$

$$[u,v]_{p,q} = \frac{\partial u}{\partial Q_n}\frac{\partial v}{\partial P_n} - \frac{\partial u}{\partial P_n}\frac{\partial v}{\partial Q_n}$$

$$\Rightarrow [u,v]_{p,q} = [u,v]_{P,Q}$$

Q. Derive Hamilton's eq. of motion in terms of Poisson bracket. imp

or

Show that, $\frac{du}{dt} = [u,H] + \frac{\partial u}{\partial t}$

where $u = u(q_s, p_s, t)$

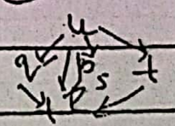
and H is Hamilton's function.

Sol:-

$$[u,H] = \frac{\partial u}{\partial q_s}\frac{\partial H}{\partial p_s} - \frac{\partial u}{\partial p_s}\frac{\partial H}{\partial q_s} \quad \text{--- (1)}$$

By Hamilton's eq. of motion

$$\dot{q}_s = \frac{\partial H}{\partial p_s} \quad \& \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} \quad \text{--- (2)}$$



Given, $u = u(q_s, p_s, t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial q_s}\frac{dq_s}{dt} + \frac{\partial u}{\partial p_s}\frac{dp_s}{dt} + \frac{\partial u}{\partial t} \quad (1)$$

$$= \frac{\partial u}{\partial q_s}\dot{q}_s + \frac{\partial u}{\partial p_s}\dot{p}_s + \frac{\partial u}{\partial t}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial q_s}\frac{\partial H}{\partial p_s} - \frac{\partial u}{\partial p_s}\frac{\partial H}{\partial q_s} + \frac{\partial u}{\partial t} \quad \text{using (2)}$$

$\Rightarrow \dot{u} = \frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$ using (1)

Show that $\dot{A} = \{A, H\} + \frac{\partial A}{\partial t}$

where $\{u, v\}$ denotes poisson bracket.

Q. Show that Lagrange's brackets are invariant under a canonical transformation (20)

Q. Using method of bilinear covariant show that the transformation

$$Q = \frac{\sqrt{2q}}{\sqrt{k}} \cos p$$

& $P = \sqrt{2q}\,\sqrt{k}\,\sin p$

is canonical.

Sol:-

As we know that a transformation is canonical if,

$$\delta P dQ - \delta Q dP = \delta p dq - \delta q dp$$

Consider,

$$dQ = \frac{1}{\sqrt{k}} \sqrt{2}\, \frac{1}{2} q^{-1/2} dq \cos p + \frac{\sqrt{2q}}{\sqrt{k}} (-\sin p)\, dp$$

$$dQ = \frac{1}{\sqrt{2k}\sqrt{q}} \cos p\, dq - \frac{\sqrt{2q}}{\sqrt{k}} \sin p\, dp$$

Now,

$$dP = \sqrt{2k}\left\{\frac{1}{2}q^{-1/2}\,dq\,\sin p\right\} + \sqrt{2q}\,\sqrt{k}\,\cos p\,dp$$

$$dP = \frac{\sqrt{k}}{\sqrt{2}\sqrt{q}}\sin p\,dq + \sqrt{2q}\,\sqrt{k}\cos p\,dp$$

Consider,

$$\delta P\,dQ - \delta Q\,dP = \left(\frac{\sqrt{k}}{\sqrt{2}\sqrt{q}}\sin p\,\delta q + \sqrt{2q}\,\sqrt{k}\cos p\,\delta p\right)$$

$$\left(\frac{1}{\sqrt{2k}\sqrt{q}}\cos p\,dq - \frac{\sqrt{2q}}{\sqrt{k}}\sin p\,dp\right)$$

$$-\left(\frac{\cos p}{\sqrt{2k}}\frac{\delta q}{\sqrt{q}} - \frac{\sqrt{2q}}{\sqrt{k}}\sin p\,\delta p\right)\left(\frac{\sqrt{k}}{\sqrt{2q}}\sin p\,dq + \sqrt{2q}\sqrt{k}\cos p\,dp\right)$$

$$= \frac{1}{2q}\sin p\cos p\,\delta q\,dq - \sin^2 p\,dp\,\delta q + \cos^2 p\,\delta p\,dq$$

$$- 2q\sin p\cos p\,\delta p\,dp - \frac{1}{2q}\sin p\cos p\,dq\,\delta q$$

$$- \cos^2 p\,dp\,\delta q + \sin^2 p\,\delta p\,dq + 2q\sin p\cos p\,\delta p\,dp$$

$$= (\cos^2 p + \sin^2 p)\,\delta p\,dq - (\cos^2 p + \sin^2 p)\,\delta q\,dp$$

$$= \delta p\,dq - \delta q\,dp$$

$$\Rightarrow \delta P\,dQ - dP\,\delta Q = \delta p\,dq - \delta q\,dp$$

Therefore transformations are canonical.

Prepaired by
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-: Pseudo or Quasi Coordinates :-

Consider the linear combination

$$\omega_s = a_{sk}\dot{q}_k \quad \text{---} (1)$$

$s, k = 1, 2, \ldots n$

We assume that relation (1) is invertible. Also let

$$\omega_s = \frac{d}{dt}(\pi_s) = \dot{\pi}_s \quad \text{---} (2)$$

π_s are called quasi coordinates. The π_s do not exist only their differentials $d\pi_s$ exist.

From (1) & (2) we obtain

$$\dot{\pi}_s = a_{sk}\dot{q}_k \quad \text{---} (3)$$

and the inverse relation is given by

$$\dot{q}_k = b_{ks}\dot{\pi}_s \quad \text{---} (4)$$

Where $a_{sk}\, b_{k\ell} = b_{k\ell}\, a_{sk} = \delta_{s\ell}$ --- (5)

Lagrange's eq. of motion in terms of Quasi coordinates.

Note:-

As, $\dot{\pi}_s = a_{sk}\dot{q}_k$

& $\dot{q}_k = b_{ks}\dot{\pi}_s$

In terms of δ variation,

$$\delta\pi_s = a_{sk}\,\delta q_k$$

& $\delta q_k = b_{ks}\,\delta\pi_s$

Three index Symbol
or
Bolzmann Symbol :-

As we know that if r is position vector of a particle then,

$$d\delta\vec{r} = \delta d\vec{r}$$

In particular,

$$d\delta q_s = \delta d q_s$$

or in general if we take an orbitarily function,

$$f = f(q_1, q_2, \ldots q_n, t)$$

$$d\delta f = \delta d f$$

Next we check whether

$$d\delta\pi_s = \delta d\pi_s$$

or not.

$\delta q_k = b_{ks}\delta\pi_s$
$dq_k = b_{ks}\, d\pi_s$
$dq_l = b_{li}\, d\pi_i$

Consider,

$$d\delta\pi_s = d\left(a_{sk}\,\delta q_k\right)$$

$$= a_{sk}\, d\delta q_k + \delta q_k \frac{\partial a_{sk}}{\partial q_l} dq_l \qquad \left[a_{sk} = a_{sk}(q_l)\right]$$

$$= a_{sk}\, d\delta q_k + \frac{\partial a_{sk}}{\partial q_l} b_{li}\, b_{kj}\, d\pi_i\, \delta\pi_j$$

$$d\delta\pi_s = a_{sk}\, d\delta q_k + \frac{\partial a_{sk}}{\partial q_l} b_{li}\, b_{kj}\, d\pi_i\, \delta\pi_j \quad \text{--- (1)}$$

Interchange d & δ in eq. (1)

$$\delta d\pi_s = a_{sk}\,\delta d q_k + \frac{\partial a_{sk}}{\partial q_l} b_{li}\, b_{kj}\, \delta\pi_i\, d\pi_j$$

Outer product
$A_{ij} B_{jkl} = A_{ikl}$

Also interchange k & l and i & j in last term

$$\delta d\pi_s = a_{sk}\,\delta dq_k + \frac{\partial a_{sl}}{\partial q_k} b_{kj}\, b_{li}\, \delta\pi_j\, d\pi_i \quad \text{--- (2)}$$

Now consider,

$$d\delta\pi_s - \delta d\pi_s = b_{li}\, b_{kj}\left(\frac{\partial a_{sk}}{\partial q_l} - \frac{\partial a_{sl}}{\partial q_k}\right) d\pi_i\, \delta\pi_j$$

If we denote

$$\gamma^s_{ij} = b_{li}\, b_{kj}\left(\frac{\partial a_{sk}}{\partial q_l} - \frac{\partial a_{sl}}{\partial q_k}\right) \quad \text{--- (*)}$$

then above expression reduces to,

$$d\delta\pi_s - \delta d\pi_s = \gamma^s_{ij}\, d\pi_i\, \delta\pi_j$$

The symbol γ^s_{ij} is called three index symbol.

Note that $d-\delta$ rule holds for qusi coordinates if $\gamma^s_{ij} = 0$ otherwise not.

Next we claim that

$$\gamma^s_{ij} = a_{sk}\left(\frac{\partial b_{ki}}{\partial q_l} b_{lj} - \frac{\partial b_{kj}}{\partial q_l} b_{li}\right)$$

As we know that

$$a_{sk}\, b_{kj} = \delta_{sj}$$

Diff partially w.r.t q_l

$$\frac{\partial a_{sk}}{\partial q_l} b_{kj} + a_{sk}\frac{\partial b_{kj}}{\partial q_l} = 0 \qquad \left(\frac{\partial \delta_{sj}}{\partial q_l} = 0\right)$$

$$\Rightarrow \frac{\partial a_{sk}}{\partial q_l} b_{kj} = -a_{sk}\frac{\partial b_{kj}}{\partial q_l} \quad \text{--- (1)}$$

(3)

Similarly if we diff.

$$a_{sl} b_{li} = \delta_{si}$$

w.r.t q_k we obtain.

$$\frac{\partial a_{sl}}{\partial q_k} b_{li} = - a_{sl} \frac{\partial b_{li}}{\partial q_k} \quad \text{——} (2)$$

From (*)

$$r_{ij}^{s} = b_{li} b_{kj} \left(\frac{\partial a_{sk}}{\partial q_l} - \frac{\partial a_{sl}}{\partial q_k} \right)$$

$$= b_{li} \left(b_{kj} \frac{\partial q_{sk}}{\partial q_l} \right) - b_{kj} \left(b_{li} \frac{\partial a_{sl}}{\partial q_k} \right)$$

$$= b_{li} \left(- a_{sk} \frac{\partial b_{kj}}{\partial q_l} \right) - b_{kj} \left(- a_{sl} \frac{\partial b_{li}}{\partial q_k} \right)$$

$$= a_{sl} b_{kj} \frac{\partial b_{li}}{\partial q_k} - a_{sk} b_{li} \frac{\partial b_{kj}}{\partial q_l}$$

Interchange l & k in first term of R.H.S.

$$= a_{sk} b_{lj} \frac{\partial b_{ki}}{\partial q_l} - a_{sk} b_{li} \frac{\partial b_{kj}}{\partial q_l}$$

$$= a_{sk} \left(\frac{\partial b_{ki}}{\partial q_l} b_{lj} - \frac{\partial b_{kj}}{\partial q_l} b_{li} \right)$$

Q. Show that r_{ij}^{s} is anti symmetric w.r.t i & j.

As $$r_{ij}^{s} = a_{sk} \left(\frac{\partial b_{ki}}{\partial q_l} b_{lj} - \frac{\partial b_{kj}}{\partial q_l} b_{li} \right)$$

$$= - a_{sk} \left(\frac{\partial b_{kj}}{\partial q_l} b_{li} - \frac{\partial b_{ki}}{\partial q_l} b_{lj} \right)$$

$$r_{ij}^{s} = - r_{ji}^{s}$$

Therefore γ_{ij} is anti symmetric.

Note:-

1. As,

$$\dot{\pi}_s = a_{sk}\,\dot{q}_k$$

2.

$$d\pi_s = a_{sk}\,dq_k$$

3.

$$\frac{\partial}{\partial \pi_s} = b_{ks}\frac{\partial}{\partial q_k}$$

4.

$$\frac{\partial}{\partial q_k} = a_{sk}\frac{\partial}{\partial \pi_s}$$

Defination:- (Lee Brackets)

The Lee brackets of X & Y is denoted and defined as (X, Y)

$$(X, Y) = XY - YX$$

Q. Show that

$$\left(\frac{\partial}{\partial \pi_s}, \frac{\partial}{\partial \pi_k}\right) = \gamma^m_{ks}\frac{\partial}{\partial \pi_m}$$

$f = g \iff f(x) = g(x) \;\forall x \in \text{Dom}$

Sol:- As,

$$\left(\frac{\partial}{\partial \pi_s}, \frac{\partial}{\partial \pi_k}\right) = \frac{\partial}{\partial \pi_s}\cdot\frac{\partial}{\partial \pi_k} - \frac{\partial}{\partial \pi_k}\frac{\partial}{\partial \pi_s} \quad \text{--- (1)}$$

Consider an orbitorly function

$$f = f(q_1, q_2, \dots, q_n, t)$$

$$\left(\frac{\partial}{\partial \pi_s}\frac{\partial}{\pi_k}\right)(f) = \frac{\partial}{\partial \pi_s}\left(\frac{\partial f}{\partial \pi_k}\right)$$

$$\left(\because \frac{\partial}{\partial \pi_s} = b_{ks}\frac{\partial}{\partial q_k}\right)$$

$$= \frac{\partial}{\partial \pi_s}\left(b_{ik}\frac{\partial f}{\partial q_i}\right)$$

$$= b_{js}\frac{\partial}{\partial q_j}\left(b_{ik}\frac{\partial f}{\partial q_i}\right)$$

$$\left(\frac{\partial}{\partial \pi_s}\frac{\partial}{\partial \pi_k}\right)(f) = b_{js}\frac{\partial b_{ik}}{\partial q_j}\frac{\partial f}{\partial q_i} + b_{js}b_{ik}\frac{\partial^2 f}{\partial q_j \partial q_i} \quad \text{--- (2)}$$

~~Similarly~~

~~$\left(\frac{\partial}{\partial \pi_k}\frac{\partial}{\partial \pi_s}\right)(f)$~~

$$= b_{js}\frac{\partial b_{ik}}{\partial q_j}a_{mi}\frac{\partial f}{\partial \pi_m} + b_{js}b_{ik}\frac{\partial^2 f}{\partial q_j \partial q_i} \quad \text{--- (3)}$$

Similarly.

$$\left(\frac{\partial}{\partial \pi_k}\frac{\partial}{\partial \pi_s}\right)(f) = b_{jk}\frac{\partial b_{is}}{\partial q_j}a_{mi}\frac{\partial f}{\partial \pi_m} + b_{jk}b_{is}\frac{\partial^2 f}{\partial q_j \partial q_i} \quad \text{--- (4)}$$

Interchanging i & j in the last term of expression (4) and then subtracting it from (3) we obtain

$$\left(\frac{\partial}{\partial \pi_s}\frac{\partial}{\partial \pi_k}\right)f - \left(\frac{\partial}{\partial \pi_k}\frac{\partial}{\partial \pi_s}\right)f = a_{mi}\left[b_{js}\frac{\partial b_{ik}}{\partial q_j} - b_{jk}\frac{\partial b_{is}}{\partial q_j}\right]\frac{\partial f}{\partial \pi_m}$$

$$\Rightarrow \left(\frac{\partial}{\partial \pi_s}, \frac{\partial}{\partial \pi_k}\right) = \gamma_{ks}^{m}\frac{\partial}{\partial \pi_m}$$

Q. If $A = A(\pi_s, t)$ then show that,

$$\dot{A} = \frac{\partial A}{\partial \pi_s}\dot{\pi}_s + \frac{\partial A}{\partial t}$$

Sol:-

Imp

Hamilton Jacobi theorem

Let $S(t, q_i, \alpha_i)$ containing n orbitarly constants be a complete solution of Hamilton Jacobi partial diff. equation

$$\frac{\partial S}{\partial t} + H\left(t, q_r, \frac{\partial S}{\partial q_r}\right) = 0$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \& \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

is obtained by solving algebric equations

$$p_i = \frac{\partial S}{\partial q_i}, \quad \beta_i = \frac{\partial S}{\partial \alpha_i}$$

where β_i are new orbittary constant.

Proof:-

As we know that in Hamilton's principle,

$$S = \int_{t_0}^{t} L\, dt$$

$$= \int_{t_0}^{t} (\dot{q}_i p_i - H)\, dt$$

$$\Rightarrow \delta S = \int_{t_0}^{t} \delta(\dot{q}_i p_i - H)\, dt$$

$$= \int_{t_0}^{t} \left[p_i \delta\dot{q}_i + \dot{q}_i \delta p_i - \left(\frac{\partial H}{\partial q_i}\delta q_i + \frac{\partial H}{\partial p_i}\delta p_i\right)\right] dt \quad \text{--- (1)}$$

Hamilton's eq.s of motion are given by,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \& \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{--- (2)}$$

From (1) & (2) we have,

$$\delta S = \int_{t_0}^{t} \left[p_i \delta \dot{q}_i + \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right] dt$$

$$\delta S = \int_{t_0}^{t} p_i \delta \dot{q}_i \, dt - \int_{t_0}^{t} \frac{\partial H}{\partial q_i} \delta q_i \, dt$$ I. BY Parts

$$= \left[p_i \delta q_i \right]_{t_0}^{t} - \int_{t_0}^{t} \dot{p}_i \delta q_i \, dt - \int_{t_0}^{t} \frac{\partial H}{\partial q_i} \delta q_i \, dt$$

$$= \left[p_i \delta q_i \right]_{t_0}^{t} - \int_{t_0}^{t} \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \, dt$$

$$= \left[p_i \delta q_i \right]_{t_0}^{t} - \int_{t_0}^{t} \left(-\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial q_i} \right) \delta q_i \, dt$$ using (2)

$$\delta S = (p_i \delta q_i)(t) - (p_i \delta q_i)(t_0)$$

$$\delta S = p_i \delta q_i - p_i^* \delta q_i^* \quad \text{——} \quad (*)$$

Where,

$$p_i^* = p_i(t_0) \quad \& \quad q_i^* = q_i(t_0)$$

$\Rightarrow$ δS contains the differentials of q_i & q_i^*

It means "S" depends on q_i & q_i^*

Therefore

$$S = S(q_i, q_i^*, t)$$

$$= S(q_i, \alpha_i, t) \quad \text{where } \alpha_i = q_i^*$$

$$\Rightarrow \delta S = \frac{\partial S}{\partial q_i} \delta q_i + \frac{\partial S}{\partial \alpha_i} \delta \alpha_i \quad \text{——} \quad (3)$$

From (*) & (3) we have,

$$p_i = \frac{\partial S}{\partial q_i} \quad (4) \qquad \& \quad -p_i^* = \frac{\partial S}{\partial \alpha_i}$$

$$\text{or} \quad \frac{\partial S}{\partial \alpha_i} = \beta_i \quad (5) \qquad \text{where } \beta_i = -p_i^*$$

As, $S = \int_{t_0}^{t} L\,dt$

$= \int_{t_0}^{t} (p_i \dot{q}_i - H)\,dt$

$\Rightarrow \frac{ds}{dt} = p_i \dot{q}_i - H$

$\Rightarrow \frac{\partial s}{\partial t} + \frac{\partial s}{\partial q_i}\dot{q}_i = p_i \dot{q}_i - H$

$\Rightarrow \frac{\partial s}{\partial t} + p_i \dot{q}_i = p_i \dot{q}_i - H$

$\Rightarrow \frac{\partial s}{\partial t} + H = 0$

$\Rightarrow \frac{\partial s}{\partial t} + H(q_i, p_i, t) = 0$

$\frac{\partial s}{\partial t} + H\left(q_i, \frac{\partial s}{\partial q_i}, t\right) = 0$

Q. Discuss the motion of projectile using Hamilton's Jacobi theory

Let a particle of unit mass is projected at an angle 'α' with velocity 'u' then the K.E of the particle is,

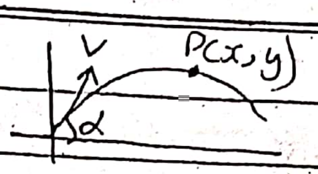
$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$= \frac{1}{2} (1) (\dot{x}^2 + \dot{y}^2)$

$\Rightarrow T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$

& $V = mgy \quad (\because m = 1)$

$V = gy$



$\Rightarrow H = T + V$

$= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + gy$ — ①

Modified Hamilton Jacobi eq. is,

The modified Hamilton Jacobi eq. is,

$H\left(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}\right) = h$ — ②

Where, $\frac{\partial S}{\partial x} = P_x$ & $\frac{\partial S}{\partial y} = P_y$

$L = T - V$

$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - gy$

$\frac{\partial L}{\partial \dot{x}} = \dot{x}, \frac{\partial L}{\partial \dot{y}} = \dot{y}$

From ① & ② we have

$\frac{1}{2}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + gy = h$ — ③

The complete solution of eq ③ is given by,

$S = \alpha x + \omega(y, \alpha, h) - ht$ — ④ Ya kya

Where H and α are constants.

From eq ④ we note that,

$\frac{\partial S}{\partial x} = \alpha \qquad \frac{\partial S}{\partial y} = \frac{\partial \omega}{\partial y}$

So eq ③ becomes,

$\frac{1}{2}\left[\alpha^2 + \left(\frac{\partial \omega}{\partial y}\right)^2\right] + gy = h$

$$\Rightarrow \left(\frac{\partial w}{\partial y}\right)^2 = 2h - 2gy - \alpha^2$$

$$= 2g\left(\frac{2h-\alpha^2}{2g} - y\right)$$

$$\left(\frac{\partial w}{\partial y}\right)^2 = 2g\left(\frac{2h-\alpha^2}{2g} - y\right)$$

$$\frac{\partial w}{\partial y} = \sqrt{2g\left(\frac{2h-\alpha^2}{2g} - y\right)}$$

$$= \sqrt{2g}\,\sqrt{k-y} \qquad \text{whe. } k = \frac{2h-\alpha^2}{2g}$$

$$\partial w = \sqrt{2g}\,\sqrt{k-y}\,\partial y$$

Integrating

$$w = \frac{-2\sqrt{2g}}{3}(k-y)^{3/2} \quad \text{(5)}$$

By Hamilton Jacobi theorem,

$$\frac{\partial S}{\partial \alpha} = -\beta \quad \text{— (6)}$$

From (4) $\quad \frac{\partial S}{\partial \alpha} = x + \frac{\partial w}{\partial \alpha} \quad$ — (7)

From (6) & (7)

$$x + \frac{\partial w}{\partial \alpha} = -\beta$$

$$\frac{\partial w}{\partial \alpha} = -(x+\beta) \quad \text{— (8)}$$

From eq (5) we have,

$$w = \frac{-2\sqrt{2g}}{3}\left[\frac{2h-\alpha^2}{2g} - y\right]^{3/2}$$

$$\Rightarrow \frac{\partial W}{\partial \alpha} = \frac{-2\sqrt{2g}}{3} \times \frac{3}{2}\left[\frac{2h-\alpha^2}{2g} - y\right]^{1/2}\left(\frac{-\alpha}{g}\right)$$

$$\Rightarrow \frac{\partial W}{\partial \alpha} = \sqrt{\frac{2}{g}}\,\alpha\,(k-y)^{1/2} \quad \text{——— (9)}$$

From (8) & (9) we have,

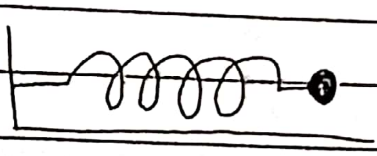
$$(x+\beta)^2 = \frac{2}{g}\alpha^2(k-y)$$

Which is the required equation of motion.

Imp. Q: Discuss the motion of one dimensional oscillator by Hamilton Jacobi method.

Sol:

Consider the motion of a one dimensional harmonic oscilator of mass "m". The restoring force is given by,



$f \propto -x$

$$f = -m\omega^{*}x \quad \text{——— (1)}$$

where ω^{*} = Spring Constant

The K.E. is given by,

$$T = \frac{1}{2}m\dot{x}^2 \quad \text{——— (2)}$$

For P.E we consider the conservative force defined in (1) as,

$$f = -m\frac{dV}{dx}$$

Force $= -\nabla V$
$= -\frac{\partial V}{\partial x} = -\frac{dV}{dx}$

$a \propto -x$
$ma \propto -xm$
$F \propto -mx$
$F = -\omega^* mx$

$\Rightarrow -m\omega^2 x = -m\frac{dV}{dx}$

$\frac{dv}{dx} = \omega^* x$

$V = \frac{\omega^* x^2}{2}$

Therefore $H = T + V$

$H = \frac{1}{2}m\dot{x}^2 + \frac{\omega^* x^2}{2}$

as $\frac{\partial H}{\partial \dot{x}} = P_x = \frac{\partial S}{\partial x}$

$m\dot{x} = \frac{\partial S}{\partial x} \Rightarrow \dot{x} = \frac{1}{m}\left(\frac{\partial S}{\partial x}\right)$

The modified Hamilton Jacobi eq. is,

$H\left(x, \frac{\partial S}{\partial x}\right) = h$

$\frac{1}{2}m\dot{x}^2 + \frac{\omega^* x^2}{2} = h$

$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\omega^* x^2}{2} = h$ — (3)

The complete solution of eq (3) is given by,

$S = -ht + w(x, h)$ — (5)

Now, $\frac{\partial S}{\partial x} = \frac{\partial w}{\partial x}$

So eq (3) becomes,

$$\frac{1}{2m}\left(\frac{\partial w}{\partial x}\right)^2 + \omega^* \frac{x^2}{2} = h$$

$$\left(\frac{\partial w}{\partial x}\right)^2 = 2mh - m\omega^* x^2$$

$$\frac{\partial w}{\partial x} = \sqrt{m\omega^*}\left(\frac{2mh}{m\omega^*} - x^2\right)^{1/2}$$

$$\Rightarrow \partial w = \sqrt{m\omega^*}\left(\frac{2h}{\omega^*} - x^2\right)^{1/2} \partial x$$

Integrating,

$$w = \sqrt{m\omega^*}\int\left(\frac{2h}{\omega^*} - x^2\right)^{1/2} \partial x$$

Put in (5)

$$s = -ht + \sqrt{m\omega^*}\int\left(\frac{2h}{\omega^*} - x^2\right)^{1/2} \partial x$$

$$\Rightarrow \frac{\partial s}{\partial h} = -t + \sqrt{m\omega^*}\int \frac{1}{2}\left(\frac{2h}{\omega^*} - x^2\right)^{-1/2}\left(\frac{2}{\omega^*}\right) \partial x$$

$$\Rightarrow \frac{\partial s}{\partial h} = -t + \sqrt{\frac{m}{\omega^*}}\int \frac{1}{\sqrt{\frac{2h}{\omega^*} - x^2}} \partial x$$

$$= -t + \sqrt{\frac{m}{\omega^*}} \sin^{-1}\left[\frac{x}{\sqrt{\frac{2h}{\omega^*}}}\right]$$

$$\frac{\partial s}{\partial h} = -t + \sqrt{\frac{m}{\omega^*}} \sin^{-1}\left(x\sqrt{\frac{\omega^*}{2h}}\right]$$

Let $\frac{\partial s}{\partial h} = t_0$

$$\Rightarrow \sqrt{\frac{m}{\omega^*}} \sin^{-1}\left[\sqrt{\frac{\omega^*}{2h}}\, x\right] = t + t_0$$

$$\sqrt{\frac{2\omega^*}{2h}}\, x = \sin\left(\sqrt{\frac{m}{\omega^*}}\,(t+t_0)\right)$$

$$x = \sqrt{\frac{2h}{\omega^*}}\, \sin\sqrt{\frac{m}{\omega^*}}\,(t+t_0)$$

Which is the eq. of S.H.M.

Q. [P#282]

Show that

$$\frac{1}{k}\left[\frac{\partial T^{(k)}}{\partial q_i^{(k)}} - (k+1)\frac{\partial T}{\partial q_i}\right] = Q_i$$

where, $T^{(k)} = \frac{d^k T}{dt^k}$ $\qquad q_i^{(k)} = \frac{d^k q_i}{dt^k}$

$i = 1, 2, \dots n$

$k = 1, 2, \dots n$

Sol:-

Consider Lagrange eq. of motion for holonomic system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad \text{--- (1)}$$

$i = 1, 2, \dots n$

where, T is K.E & Q_i is generalized force corresponding to the coordinates q_i

As, $T = T(q_i, \dot{q}_i, t)$

Diff. w.r.t 't'

$$\dot{T} = \frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial T}{\partial t} \quad \text{--- (2)}$$

Diff. w.r.t $\dot{q}_i$ (i replace j)

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} = \frac{\partial T}{\partial q_i} + \frac{\partial^2 T}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial t} \quad \text{--- (3)}$$

As we know that,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial^2 T}{\partial q_j \partial \dot{q}_i}\dot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial t} +$$

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Using in ③

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} = \frac{\partial T}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) \quad \text{—— (4)}$$

From ① & ④

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} = \frac{\partial T}{\partial q_i} + Q_i + \frac{\partial T}{\partial q_i}$$

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} = 2\frac{\partial T}{\partial q_i} + Q_i$$

$$\left[\frac{\partial \dot{T}}{\partial \dot{q}_i} - 2\frac{\partial T}{\partial q_i}\right] = Q_i \quad \text{—— (5)}$$

As,

$$\dot{T} = \frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial T}{\partial t}$$

Diff. w.r.t 't'

$$\ddot{T} = \frac{\partial T}{\partial q_i}\ddot{q}_i + \frac{\partial^2 T}{\partial q_j \partial q_i}\dot{q}_j \dot{q}_i + \frac{\partial^2 T}{\partial \dot{q}_j \partial q_i}\ddot{q}_j \dot{q}_i$$

$$+ \frac{\partial^2 T}{\partial t \partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\dddot{q}_i + \frac{\partial^2 T}{\partial q_j \partial \dot{q}_i}\dot{q}_j \ddot{q}_i$$

$$+ \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_j \ddot{q}_i + \frac{\partial^2 T}{\partial t \partial \dot{q}_i}\ddot{q}_i$$

$$+ \frac{\partial^2 T}{\partial q_j \partial t}\dot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_j \partial t}\ddot{q}_j + \frac{\partial^2 T}{\partial t^2}$$

$$\dddot{T} = \frac{\partial T}{\partial q_i}\dddot{q}_i + 2\frac{\partial^2 T}{\partial \dot{q}_j \partial q_i}\ddot{q}_j \dot{q}_i + \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_j \ddot{q}_i$$

$$+ 2\frac{\partial^2 T}{\partial \dot{q}_j \partial t}\ddot{q}_j + \text{terms not containing } \ddot{q}_i$$

Diff. w.r.t $\ddot{q}_i$

$$\frac{\partial \ddot{T}}{\partial \ddot{q}_i} = \frac{\partial T}{\partial q_i} + 2\frac{\partial^2 T}{\partial \dot{q}_i \partial q_i}\dot{q}_i + 2\frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_j + 2\frac{\partial^2 T}{\partial \dot{q}_i \partial t}$$

$$= \frac{\partial T}{\partial q_i} + 2\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$$

$$= \frac{\partial T}{\partial q_i} + 2\left\{Q_i + \frac{\partial T}{\partial q_i}\right\} \quad \text{from (1)}$$

$$= 3\frac{\partial T}{\partial q_i} + 2Q_i$$

$$\left[\frac{\partial \ddot{T}}{\partial \ddot{q}_i} - 3\frac{\partial T}{\partial q_i}\right] = 2Q_i$$

$$\Rightarrow \frac{1}{2}\left[\frac{\partial \ddot{T}}{\partial \ddot{q}_i} - 3\frac{\partial T}{\partial q_i}\right] = Q_i \quad \text{---(6)}$$

From eq (5)

$$\frac{1}{1}\left[\frac{\partial \dot{T}}{\partial \dot{q}_i} - (1+1)\frac{\partial T}{\partial q_i}\right] = Q_i$$

From eq (6)

$$\frac{1}{2}\left[\frac{\partial \ddot{T}}{\partial \ddot{q}_i} - (2+1)\frac{\partial T}{\partial q_i}\right] = Q_i$$

⋮

$$\frac{1}{k}\left[\frac{\partial T^{(k)}}{\partial q_i^{(k)}} - (k+1)\frac{\partial T}{\partial q_i}\right] = Q_i$$

As required

(4)

Q. Show that total energy of a system is constant

Sol:-

As we know that eq. of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0$$

or

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_s}(T-V)\right\} - \frac{\partial}{\partial q_s}(T-V) = 0$$

$$\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_s} - 0\right\} - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = 0 \quad \text{— (1)}$$

Also we know that K.E. of a system is given by,

$$2T = \dot{q}_s \frac{\partial T}{\partial \dot{q}_s}$$

Diff. w.r.t "t"

$$\frac{d}{dt}(2T) = \dot{q}_s \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) + \frac{\partial T}{\partial \dot{q}_s}\frac{d}{dt}(\dot{q}_s)$$

$$\frac{d}{dt}(2T) = \dot{q}_s \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) + \frac{\partial T}{\partial \dot{q}_s}\ddot{q}_s$$

$$\Rightarrow \quad \dot{q}_s \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) = \frac{d}{dt}(2T) - \frac{\partial T}{\partial \dot{q}_s}\ddot{q}_s \quad \text{— (2)}$$

Multiplying eq (1) by $\dot{q}_s$

$$\dot{q}_s \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \dot{q}_s \frac{\partial T}{\partial q_s} + \dot{q}_s \frac{\partial V}{\partial q_s} = 0$$

$$\Rightarrow \quad \frac{d}{dt}(2T) - \frac{\partial T}{\partial \dot{q}_s}\ddot{q}_s - \dot{q}_s \frac{\partial T}{\partial q_s} + \dot{q}_s \frac{\partial V}{\partial q_s} = 0 \quad \text{— (3)}$$

using (2)

Also, $T = T(q_s, \dot{q}_s)$ (we consider stationary (time ind. state) state)

& $V = V(q_s)$

$$\left.\begin{aligned} \therefore \quad \frac{dT}{dt} &= \frac{\partial T}{\partial q_s}\dot{q}_s + \frac{\partial T}{\partial \dot{q}_s}\ddot{q}_s \\ \& \quad \frac{dV}{dt} &= \frac{\partial V}{\partial q_s}\dot{q}_s \end{aligned}\right\} \quad \text{——— (4)}$$

Eq. (3) can be written as,

$$2\frac{dT}{dt} - \left(\frac{\partial T}{\partial \dot{q}_s}\ddot{q}_s + \dot{q}_s\frac{\partial T}{\partial q_s}\right) + \dot{q}_s\frac{\partial V}{\partial q_s} = 0$$

$$\Rightarrow 2\frac{dT}{dt} - \frac{dT}{dt} + \frac{dV}{dt} = 0 \qquad \text{using (4)}$$

$$\frac{d}{dt}(T+V) = 0$$

$$\Rightarrow T+V = \text{Constant}$$

-: First Integral :-

Consider the function,

$$f = f(q_s, \dot{q}_s, t) \qquad s = 1, 2, \dots n$$

$$\frac{df}{dt} = \frac{\partial f}{\partial q_s}\dot{q}_s + \frac{\partial f}{\partial \dot{q}_s}\ddot{q}_s + \frac{\partial f}{\partial t} \quad (1)$$

The function f is called first integral if $\frac{df}{dt} = 0$

or $f = \text{Constant}$ [f is called first integral or constant of motion]

:Energy Integral:-

If Lagrangian "L" is not an explicit function of "t" i.e.

$$L = L(q_s, \dot{q}_s)$$

then, $$\frac{dL}{dt} = \frac{\partial L}{\partial q_s}\dot{q}_s + \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s \quad \text{——} (1)$$

Also we know that,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0$$

$$\frac{\partial L}{\partial q_s} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)$$

Put in (1)

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\dot{q}_s + \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}\right)$$

$$\frac{d}{dt}\left\{L - \dot{q}_s\frac{\partial L}{\partial \dot{q}_s}\right\} = 0$$

$$\Rightarrow \quad L - \dot{q}_s\frac{\partial L}{\partial \dot{q}_s} = \text{Constant} = h \quad \text{(Planck's const.)}$$

$$L - \dot{q}_s\frac{\partial L}{\partial \dot{q}_s} = h$$

$$\Rightarrow \quad L - \dot{q}_s\frac{\partial L}{\partial \dot{q}_s} = h \quad \text{——} (2)$$

This is called energy integral

Also we know that for stationary coordinate system,

$$\frac{\partial T}{\partial \dot{q}_s}\dot{q}_s = \frac{\partial L}{\partial \dot{q}_s}\dot{q}_s = 2T$$

$$\dot{q}_s\frac{\partial L}{\partial \dot{q}_s} = \dot{q}_s\frac{\partial}{\partial \dot{q}_s}(T-V) = \dot{q}_s\frac{\partial T}{\partial \dot{q}_s} - 0 = \dot{q}_s\frac{\partial T}{\partial \dot{q}_s} = 2T$$

So eq. (2) becomes,

$$L - 2T = h$$

$$T - V - 2T = h$$

$$T + V = -h$$

$$T + V = \text{Constant}$$

Q. Define energy integral and constant of motion, also prove law of conservation of energy using energy integral.

Q. Show that Hamiltonian function H is equal to the total energy.

i.e. $H = T + V$

Sol:-

By Euler's eq.

$$2T = \dot{q}_s\frac{\partial L}{\partial \dot{q}_s}$$

but, $\frac{\partial L}{\partial \dot{q}_s} = p_s$

$$\Rightarrow \quad 2T = \dot{q}_s p_s \quad \text{--- (1)}$$

As,

$$H = \dot{q}_s p_s - L$$

$$H = 2T - L \quad \text{using (1)}$$

$$H = 2T - (T - V)$$

$$\boxed{H = T + V}$$

111

Imp

Q. A particle of mass "m" is attached to a point O by inverse square force,

$$F = \frac{-\mu m}{r^2}$$

Where μ is gravitational coefficient using plane polar coordinates (r, θ) find Hamilton's eq. of motion

Sol:-

As, $F = \frac{-\mu m}{r^2}$

but, $F = -\frac{\partial V}{\partial r}$ (by def. of conservative force)

$$\Rightarrow -\frac{\partial V}{\partial r} = -\frac{\mu m}{r^2}$$

$$\partial V = \frac{\mu m}{r^2}\partial r$$

$$\Rightarrow V = -\frac{\mu m}{r}$$

Also we know that the K.E in terms of polar coordinates is given by;

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$\because L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu m}{r}$$

As, $H = \dot{q}_s p_s - L$

$$= \dot{r}p_r + \dot{\theta}p_\theta - L$$

$$H = \dot{r}p_r + \dot{\theta}p_\theta - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu m}{r} \quad \text{---} (1)$$

As, $\frac{\partial L}{\partial \dot{r}} = p_r$

$m\dot{r} = p_r \Rightarrow \dot{r} = \frac{p_r}{m}$

& $\frac{\partial L}{\partial \dot{\theta}} = p_\theta$

$mr^2\dot{\theta} = p_\theta \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$

So eq. (*) becomes,

$$H = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{1}{2}m\left(\frac{p_r^2}{m^2} + \frac{r^2 p_\theta^2}{m^2 r^4}\right) - \frac{\mu m}{r}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{\mu m}{r}$$

The corresponding Hamilton's eq. of motion are.

$$\frac{\partial H}{\partial r} = -\dot{p}_r \quad , \quad \frac{\partial H}{\partial p_r} = \dot{r} \quad \text{——— (1)}$$

& $$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad , \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \text{——— (2)}$$

Now,

$$\frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{\mu m}{r^2}$$

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\frac{\partial H}{\partial \theta} = 0 \quad , \quad \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

So ① & ② becomes,

$$\frac{-p_\theta^2}{mr^3} + \frac{\mu m}{r^2} = -\dot{p}_r \quad , \quad \frac{p_r}{m} = \dot{r} \quad —③$$

$$\dot{p}_\theta = 0 \quad , \quad \frac{p_\theta}{mr^2} = \dot{\theta} \quad —④$$

Eq.s ③ and ④ are required Hamilton's eq. of motion.

Q. Find the Lagrange's eq. of motion using plane polar coordinates & inverse square law.

Q. Identify ignorable & non ignorable coord. in above problem & find corresponding Rothean eq. of motion for non ignorable coordinates.

Q. Derive Lagrange's eq. of motion in terms of

(i) Plane polar coordinates
(ii) Spherical polar coordinates
(iii) Cylindrical coordinates

Q. Find the Hamilton's eq. of motion in terms of

(i) (ii) (iii) above

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Q. Find

(i) Lagrange's eq. of motion

(ii) Hamilton's eq. of motion

if $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$

& $V = l(1 + \cos\theta)$

Note:- The Lagrange's eq. of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0$$

or $E(L) = 0$

where, $E = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_s}\right) - \frac{\partial}{\partial q_s}$

E is called Euler operator

-: Gauge Variate :-

If the Lagrangian function "L" under the transformation,

$$\bar{t} = t + \epsilon\,\phi_s$$

$$\bar{q}_s = q_s + \epsilon\,\psi_s \qquad F = F(q_s, \dot{q}_s, t)$$

satisfy the relation

$$L(\bar{q}_s, \dot{\bar{q}}_s, \bar{t})\,d\bar{t} - L(q_s, \dot{q}_s, t)\,dt = \epsilon\,dF$$

then "L" is called Gauge variate.

GENTLEMAN

Q.

Given,

x	0	1	3	4	6
y	-2	8	10	15	13

calculate $y(2.6)$

Sol:-

$$y(2.6) = \frac{(2.6-1)(2.6-3)(2.6-4)(2.6-6)}{(0-1)(0-3)(0-4)(0-6)}(-2)$$

$$+ \frac{(2.6-0)(2.6-3)(2.6-4)(2.6-6)}{(1-0)(1-3)(1-4)(1-6)}(8)$$

$$+ \frac{(2.6-0)(2.6-1)(2.6-4)(2.6-6)}{(3-0)(3-1)(3-4)(3-6)}(10)$$

$$+ \frac{(2.6-0)(2.6-1)(2.6-3)(2.6-6)}{(4-0)(4-1)(4-3)(4-6)}(15)$$

$$+ \frac{(2.6-0)(2.6-1)(2.6-3)(2.6-4)}{(6-0)(6-1)(6-3)(6-4)}(13)$$

Imp

-: Invariance Identity :-

If "L" is a Gauge Variate under the transformation,

$$\bar{t} = t + \epsilon \phi_s$$

$$\bar{q}_s = q_s + \epsilon \psi_s$$

then show that the Lagrangian function satisfy the identity,

$$\left(\phi_s \frac{\partial L}{\partial t} + \psi_s \frac{\partial L}{\partial q_s}\right) + \left(\dot{\psi}_s - \dot{\phi}_s \dot{q}_s\right) \frac{\partial L}{\partial \dot{q}_s} + L\dot{\phi}_s = \frac{dF}{dt}$$

This identity is known as invariance identity.

Sol:-

As "L" is a Gauge variate so it satisfies the relation,

$$L(\bar{q}_s, \dot{\bar{q}}_s, t)\, d\bar{t} - L(q_s, \dot{q}_s, t)\,dt = \epsilon\, dF$$

Dividing by "dt"

$$L(\bar{q}_s, \dot{\bar{q}}_s, t) \frac{d\bar{t}}{dt} - L(q_s, \dot{q}_s, t) = \epsilon \frac{dF}{dt}$$

Diff. w.r.t ϵ

$$\frac{d}{d\epsilon}\left(L(\bar{q}_s, \dot{\bar{q}}_s, \bar{t})\right) \frac{d\bar{t}}{dt} + L(\bar{q}_s, \dot{\bar{q}}_s, \bar{t}) \frac{d}{d\epsilon}\left(\frac{d\bar{t}}{dt}\right)$$

$$- \frac{d}{d\epsilon} L(q_s, \dot{q}_s, t) = \frac{dF}{dt} \quad \text{———} \quad (*)$$

As, $\bar{t} = t + \epsilon \phi_s$

Diff. w.r.t "t"

$$\frac{d\bar{t}}{dt} = 1 + \epsilon \dot{\phi}_s \quad (\because \epsilon \text{ is ind. of } t)$$

Diff. w.r.t ϵ

$$\frac{d}{d\epsilon}\left(\frac{d\bar{t}}{dt}\right) = \dot{\phi}_s \quad \text{—— (1)}$$

Also note that $\left(\frac{d\bar{t}}{dt}\right)_{\epsilon=0} = 1$

Also,

$$\bar{q}_s = q_s + \epsilon \psi_s$$

$$\Rightarrow \frac{d\bar{q}_s}{d\bar{t}} = \frac{d}{d\bar{t}}(q_s + \epsilon\psi_s)$$

$$\Rightarrow \dot{\bar{q}}_s = \frac{d}{dt}(q_s + \epsilon\psi_s)\frac{dt}{d\bar{t}}$$

$$= (\dot{q}_s + \epsilon\dot{\psi}_s)\left(\frac{1}{1+\epsilon\dot{\phi}_s}\right)$$

$$\dot{\bar{q}}_s = (\dot{q}_s + \epsilon\dot{\psi}_s)(1+\epsilon\dot{\phi}_s)^{-1}$$

$$= (\dot{q}_s + \epsilon\dot{\psi}_s)(1-\epsilon\dot{\phi}_s) \quad \text{(using binomial series \& neglecting)}$$

$$= \dot{q}_s - \dot{q}_s\epsilon\dot{\phi}_s + \epsilon\dot{\psi}_s - \epsilon^2\dot{\psi}_s\dot{\phi}_s$$

$$\dot{\bar{q}}_s = \dot{q}_s + \epsilon\dot{\psi}_s - \epsilon\dot{q}_s\dot{\phi}_s$$

Consider

$$L(\bar{q}_s, \dot{\bar{q}}_s, \bar{t}) = L(q_s + \epsilon\psi_s,\ \dot{q}_s + \epsilon\dot{\psi}_s - \epsilon\dot{q}_s\dot{\phi}_s,\ t + \epsilon\phi_s)$$

$$= L(q_s, \dot{q}_s, t) + \epsilon\psi_s\frac{\partial L}{\partial q_s} + (\epsilon\dot{\psi}_s - \epsilon\dot{q}_s\dot{\phi}_s)\frac{\partial L}{\partial \dot{q}_s} + \epsilon\phi_s\frac{\partial L}{\partial t}$$

(using Taylor's theorem)

Diff w.r.t ϵ

$$\frac{d}{d\epsilon}\left(L(\bar{q}_s, \dot{\bar{q}}_s, \bar{t})\right) = 0 + \psi_s\frac{\partial L}{\partial q_s} + (\dot{\psi}_s - \dot{q}_s\dot{\phi}_s)\frac{\partial L}{\partial \dot{q}_s} + \phi_s\frac{\partial L}{\partial t} \quad \text{—— (2)}$$

using (1) & (2) in (*)

$$\left[\psi_s\frac{\partial L}{\partial q_s} + (\dot{\psi}_s - \dot{q}_s\dot{\phi}_s)\frac{\partial L}{\partial \dot{q}_s} + \phi_s\frac{\partial L}{\partial t}\right](1+\epsilon\dot{\phi}_s)$$

$$+ \left[L(q_s, \dot{q}_s, t) + \epsilon\psi_s\frac{\partial L}{\partial q_s} + (\epsilon\dot{\psi}_s - \epsilon\dot{q}_s\dot{\phi}_s)\frac{\partial L}{\partial \dot{q}_s} + \epsilon\phi_s\frac{\partial L}{\partial t}\right]\frac{d}{d\epsilon}(1+\epsilon\dot{\phi}_s)$$

Imp

-: Noether's theorem :-

If the Lagrangian "L" is a gage variate under the transformation,

$$\bar{t} = t + \epsilon \phi_s$$
$$\bar{q}_s = q_s + \epsilon \psi_s$$

then

$$\phi_s L + (\psi_s - \phi_s \dot{q}_s) \frac{\partial L}{\partial \dot{q}_s} - F$$

is a constant of motion.

Proof:-

We have to prove,

$$\frac{d}{dt}\left[\phi_s L + (\psi_s - \phi_s \dot{q}_s) \frac{\partial L}{\partial \dot{q}_s} - F\right] = 0$$

As L is a Gauge variate, therefore it must satisfies the invariant identity i.e.

$$\left(\phi_s \frac{\partial L}{\partial t} + \psi_s \frac{\partial L}{\partial q_s}\right) + \frac{\partial L}{\partial \dot{q}_s}(\dot{\psi}_s - \dot{\phi}_s \dot{q}_s) + L\dot{\phi}_s = \frac{dF}{dt}$$

$$\Rightarrow \phi_s \frac{\partial L}{\partial t} + \psi_s \frac{\partial L}{\partial q_s} + \dot{\psi}_s \frac{\partial L}{\partial \dot{q}_s} - \dot{\phi}_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} + L\dot{\phi}_s = \frac{dF}{dt} \quad \text{—ⓐ}$$

As, $L = L(q_s, \dot{q}_s, t)$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_s}\dot{q}_s + \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{\partial L}{\partial t} = \frac{dL}{dt} - \frac{\partial L}{\partial q_s}\dot{q}_s - \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s$$

Multiplying with ϕ_s

$$\phi_s \frac{dL}{dt} - \frac{\partial L}{\partial q_s}\dot{q}_s\phi_s - \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s\phi_s = \frac{\partial L}{\partial t}\phi_s \quad \text{--- (1)}$$

Now consider

$$\frac{d}{dt}\left(\phi_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s}\right) = \phi_s \dot{q}_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) + \phi_s \ddot{q}_s \frac{\partial L}{\partial \dot{q}_s} + \dot{\phi}_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s}$$

$$\Rightarrow \dot{\phi}_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} = \frac{d}{dt}\left(\phi_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s}\right) - \phi_s \dot{q}_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \phi_s \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s \quad \text{--- (2)}$$

Now consider

$$\frac{d}{dt}\left(\psi_s \frac{\partial L}{\partial \dot{q}_s}\right) = \psi_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) + \dot{\psi}_s \frac{\partial L}{\partial \dot{q}_s}$$

$$\Rightarrow \dot{\psi}_s \frac{\partial L}{\partial \dot{q}_s} = \frac{d}{dt}\left(\psi_s \frac{\partial L}{\partial \dot{q}_s}\right) - \psi_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) \quad \text{--- (3)}$$

Put eq. (1), (2) & (3) in (a)

$$\phi_s \frac{dL}{dt} - \frac{\partial L}{\partial q_s}\dot{q}_s\phi_s - \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s\phi_s + \psi_s \frac{\partial L}{\partial q_s}$$

$$+ \frac{d}{dt}\left(\psi_s \frac{\partial L}{\partial \dot{q}_s}\right) - \psi_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)$$

$$- \frac{d}{dt}\left(\phi_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s}\right) + \phi_s \dot{q}_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) + \phi_s \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s$$

$$+ L\dot{\phi}_s - \frac{dF}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}\left[\phi_s L + \psi_s \frac{\partial L}{\partial \dot{q}_s} - \phi_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} - F\right] = 0$$

Jacobi Identity

$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$

$[u,v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$

$\frac{\partial H}{\partial p_i} = \dot{q}_i$ & $\frac{\partial H}{\partial q_i} = -\dot{p}_i$

-: Poisson Theorem :-

The poisson bracket of two constants of motion is itself a constant of motion.

or

If $u(q_i, p_i, t)$ & $v(q_i, p_i, t)$ are two integrals of a canonical equation then, $[u,v]$ is third integral of these equations.

Proof:-

As we know that Hamilton's eq.s of motion are,

$$① \text{---} \frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \& \quad \frac{\partial H}{\partial p_i} = \dot{q}_i \text{---} ②$$

Since u is a contant of motion

$$\Rightarrow \frac{du}{dt} = 0$$

$$\Rightarrow \frac{\partial u}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial u}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt} = 0$$

$$\frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t} = 0$$

using ① & ② we have,

$$\frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t} = 0$$

$$[u,H] + \frac{\partial u}{\partial t} = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} = -[u,H] \text{---} ③$$

Similarly,

$$\frac{\partial V}{\partial t} = -[V,H] \quad \text{—— (4)}$$

($\because$ V is also const. of motion)

To prove $[u,v]$ is a constant of motion we have to prove that,

$$\frac{\partial}{\partial t}[u,v] + [[u,v],H] = 0$$

Consider;

$$\frac{\partial}{\partial t}[u,v] + [[u,v],H]$$

$$= \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] + [[u,v],H]$$

$$= \left[-[u,H],v\right] + \left[u,-[v,H]\right] + [[u,v],H] \quad \text{using (3) \& (4)}$$

$$= -[H,[u,v]] - [[u,H],v] - [u,[v,H]]$$

$$= -[H,[u,v]] - [u,[v,H]] - [v,[H,u]]$$

$$= -\{[H,[u,v]] + [u,[v,H]] + [v,[H,u]]\}$$

$$= 0 \quad \text{(using Jacobi identity)}$$

$$= R.H.S$$

$\therefore \Rightarrow$ $[u,v]$ is a constant of motion.

$L(q_s, \dot{q}_s, t)$ 122

$R(q_j, \dot{q}_j, p_i, t)$

(A 2011) $H(q_i, p_i, t)$

:- Whittaker's Equations of Motion :-

Let us consider a dynamical system with "n" generalized coordinates q_s $(s = 1, 2, 3, \dots n)$ generalized velocities $\dot{q}_s$ and Lagrangian $L(q_s, \dot{q}_s)$ then the corresponding Lagrange's eq. of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \quad \text{--- (1)}$$

$$s = 1, 2, 3, \dots n$$

The energy integral is given by

$$\dot{q}_s \frac{\partial L}{\partial \dot{q}_s} - L = \text{constant} \quad \text{--- (2)}$$

(1) Consider

$$\frac{d}{dt}\left\{\dot{q}_s \frac{\partial L}{\partial \dot{q}_s} - L\right\} = \ddot{q}_s \frac{\partial L}{\partial \dot{q}_s} + \dot{q}_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s}\dot{q}_s - \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s$$

$$= \dot{q}_s\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s}\right\}$$

$$= 0 \quad \text{using (1)}$$

$$\Rightarrow \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} - L = \text{Constant}$$

Now let us sppose that,

$$\frac{dq_i}{dt} = \dot{q}_i$$

$$\frac{dq_i}{dq_i}\frac{dq_i}{dt} = \dot{q}_i$$

$$q_i' = \frac{\dot q_i}{\dot q_1}, \qquad \frac{\partial q_i'}{\partial \dot q_i} = \frac{1}{\dot q_1}$$

$$q_i' = \frac{\dot q_i}{\dot q_1} = \dot q_i \dot q_1^{-1}, \qquad \frac{\partial q_i'}{\partial \dot q_1} = -\dot q_i \dot q_1^{-2} = -\frac{\dot q_i}{\dot q_1^2}$$

$$q_i' \dot q_1 = \dot q_i \qquad i = 2, 3, 4, \dots n$$

$$\Rightarrow q_i' = \frac{\dot q_i}{\dot q_1} \quad \text{——} (*)$$

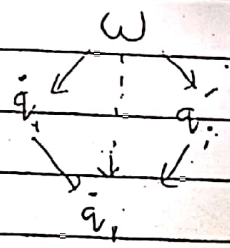
Now make this change in Lagrangian function L and denote new funtion by ω

$$\therefore L(q_s, \dot q_s) = \omega(q_s, \dot q_1, q_i')$$

$$s = 1, 2, 3, \dots n$$
$$i = 2, 3, 4, \dots n$$

Consider,

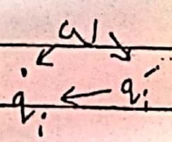
$$\frac{\partial L}{\partial \dot q_1} = \frac{\partial \omega}{\partial \dot q_1}\frac{\partial \dot q_1}{\partial \dot q_1} + \frac{\partial \omega}{\partial q_i'}\frac{\partial q_i'}{\partial \dot q_1}$$



$$= \frac{\partial \omega}{\partial \dot q_1} - \frac{\dot q_i}{\dot q_1^2}\cdot\frac{\partial \omega}{\partial q_i'} \quad \text{using } (*)$$

$$\Rightarrow \frac{\partial L}{\partial \dot q_1} = \frac{\partial \omega}{\partial \dot q_1} - \frac{\dot q_i}{\dot q_1^2}\frac{\partial \omega}{\partial q_i'} \quad \text{——} (3)$$

Now consider,



$$\frac{\partial L}{\partial \dot q_i} = \frac{\partial \omega}{\partial q_i'}\frac{\partial q_i'}{\partial \dot q_i}$$

$$\frac{\partial L}{\partial \dot q_i} = \frac{1}{\dot q_1}\frac{\partial \omega}{\partial q_i'} \quad \text{——} (4)$$

Now,

$$\frac{\partial L}{\partial q_s} = \frac{\partial \omega}{\partial q_s} \quad \text{——} (5)$$

Now consider the energy integral for q_1,

$$\dot q_1 \frac{\partial \omega}{\partial \dot q_1} - \omega = h \text{ (constant)} \quad \text{——} (A)$$

$w(q_s, \dot{q}_1, q_i') \Rightarrow \frac{\partial w}{\partial \dot{q}_1}(q_s, \dot{q}_1, q_i')$

$\frac{\partial w}{\partial \dot{q}_1}$: $\swarrow \dot{q}_1$, $\searrow q_i'$; $\dot{q}_1 \to q_i' \leftarrow q_i'$

Diff. eq (A) w.r.t q_i'

$$\dot{q}_1 \left\{ \frac{\partial^2 w}{\partial \dot{q}_1^2} \frac{\partial \dot{q}_1}{\partial q_i'} + \frac{\partial^2 w}{\partial q_i' \partial \dot{q}_1} \frac{\partial q_i'}{\partial q_i'} \right\} + \frac{\partial w}{\partial \dot{q}_1} \frac{\partial \dot{q}_1}{\partial q_i'}$$

$$- \frac{\partial w}{\partial \dot{q}_1} \frac{\partial \dot{q}_1}{\partial q_i'} - \frac{\partial w}{\partial q_i'}(1) = 0$$

$$\dot{q}_1 \frac{\partial^2 w}{\partial \dot{q}_1^2} \frac{\partial \dot{q}_1}{\partial q_i'} + \dot{q}_1 \frac{\partial^2 w}{\partial q_i' \partial \dot{q}_1} + \frac{\partial w}{\partial \dot{q}_1} \frac{\partial \dot{q}_1}{\partial q_i'} - \frac{\partial w}{\partial \dot{q}_1} \frac{\partial \dot{q}_1}{\partial q_i'} - \frac{\partial w}{\partial q_i'} = 0$$

$$\Rightarrow \dot{q}_1 \frac{\partial^2 w}{\partial \dot{q}_1^2} \frac{\partial \dot{q}_1}{\partial q_i'} + \dot{q}_1 \frac{\partial^2 w}{\partial q_i' \partial \dot{q}_1} - \frac{\partial w}{\partial q_i'} = 0 \quad \text{—(6)}$$

Now diff. eq (A) w.r.t q_s

$$\dot{q}_1 \frac{\partial^2 w}{\partial q_s \partial \dot{q}_1} - \frac{\partial w}{\partial q_s} = 0 \quad \text{—(7)}$$

Now let, $L' = \frac{\partial w}{\partial \dot{q}_1}$ (8)

$$\Rightarrow \frac{\partial L'}{\partial q_i'} = \frac{\partial^2 w}{\partial q_i' \partial \dot{q}_1} + \frac{\partial^2 w}{\partial \dot{q}_1^2} \frac{\partial \dot{q}_1}{\partial q_i'}$$

$\frac{\partial w}{\partial \dot{q}_1}(q_s, \dot{q}_1, q_i')$: $\swarrow \dot{q}_1$, $\searrow q_i'$; $\searrow q_i' \swarrow$

and,

$$\frac{\partial L'}{\partial q_s} = \frac{\partial^2 w}{\partial q_s \partial \dot{q}_1} \quad \text{—(9)}$$

From eq. (6)

$$\dot{q}_1 \left[\frac{\partial^2 w}{\partial \dot{q}_1^2} \frac{\partial \dot{q}_1}{\partial q_i'} + \frac{\partial^2 w}{\partial q_i' \partial \dot{q}_1} \right] - \frac{\partial w}{\partial q_i'} = 0$$

$$\dot{q}_1 \frac{\partial L'}{\partial q_i'} - \frac{\partial w}{\partial q_i'} = 0 \quad \text{using (8)}$$

$$\Rightarrow \frac{\partial L'}{\partial q_i'} = \frac{1}{\dot{q}_1} \frac{\partial w}{\partial q_i'} \quad \text{—(10)}$$

(12a)

From eq. (7) & (9) we have,

$$\dot{q}_1 \frac{\partial L'}{\partial q_s} - \frac{\partial \omega}{\partial q_s} = 0$$

$$\dot{q}_1 \frac{\partial L'}{\partial q_s} - \frac{\partial \omega}{\partial q_s} = 0$$

$$\Rightarrow \frac{\partial L'}{\partial q_s} = \frac{1}{\dot{q}_1} \frac{\partial \omega}{\partial q_s} \quad \text{--- (11)}$$

From eq. (4) & (10) we have,

$$\frac{\partial L'}{\partial q_i'} = \frac{\partial L}{\partial \dot{q}_i} \quad \text{--- (12)}$$

From eq. (4)

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{1}{\dot{q}_1} \frac{\partial \omega}{\partial q_i'}$$

From eq. (5) & (11) we have,

$$\dot{q}_1 \frac{\partial L'}{\partial q_s} = \frac{\partial L}{\partial q_s} \quad \text{--- (13)}$$

From eq (1)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial q_i'}\right) - \dot{q}_1 \frac{\partial L'}{\partial q_i} = 0 \quad \text{using (12) \& (13)}$$

$$\frac{d}{dq_1}\left(\frac{\partial L'}{\partial q_i'}\right) \frac{dq_r}{dt} - \dot{q}_1 \frac{\partial L'}{\partial q_i} = 0$$

$$\frac{d}{dq_1}\left(\frac{\partial L'}{\partial q_i'}\right) \dot{q}_1 - \dot{q}_i \frac{\partial L'}{\partial q_i} = 0$$

or

$$\frac{d}{dq_1}\left(\frac{\partial L'}{\partial q_i'}\right) - \frac{\partial L'}{\partial q_i} = 0$$

This is known as Wrtteker's eq. of motion

Kinetic Energy of a Dynamical System

(In terms of generalized coordinates)

Let us consider a dynamical system of "n" particles. If at any time "t" radius vector of the ith vector of mass "m_i" is "r_i" and having velocity "v_i" then the expression for the K.E denoted by "T" is,

$$T = \frac{1}{2} m_i v_i^2$$

$$= \frac{1}{2} m_i \dot{r}_i^2 \quad \text{—— (1)}$$

To express the K.E of the system in terms of generalized coordinates we substitute,

$$\dot{r}_i = \frac{\partial r_i}{\partial q_s} \dot{q}_s + \frac{\partial r_i}{\partial t}$$

So, $$T = \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial q_s} \dot{q}_s + \frac{\partial r_i}{\partial t} \right) \left(\frac{\partial r_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t} \right)$$

$s = 1, 2, 3, \ldots n$
$k = 1, 2, 3, \ldots n$

$$\Rightarrow T = \frac{1}{2} m_i \left[\frac{\partial r_i}{\partial q_s} \frac{\partial r_i}{\partial q_k} \dot{q}_s \dot{q}_k + \left(\frac{\partial r_i}{\partial t} \right)^2 + 2 \frac{\partial r_i}{\partial q_s} \frac{\partial r_i}{\partial t} \dot{q}_s \right]$$

Put, $$a_{sk} = m_i \frac{\partial r_i}{\partial q_s} \frac{\partial r_i}{\partial q_k}$$

$$a_s = m_i \frac{\partial r_i}{\partial q_s} \frac{\partial r_i}{\partial t} \quad \& \quad a_o = \left(\frac{\partial r_i}{\partial t} \right)^2 m_i$$

So, $$T = \frac{1}{2} \left(a_{sk} \dot{q}_s \dot{q}_k + 2 a_s \dot{q}_s + a_o \right)$$

where a_{sk}, a_s, a_o are the functions of q_s and t

Let, $T_2 = \frac{1}{2} a_{sk} \dot{q}_s \dot{q}_k$

$T_1 = \frac{1}{2} a_s \dot{q}_s$ & $T_o = \frac{1}{2} a_o$

Kinetic Energy of a Dynamical System

(In terms of generalized coordinates)

Let us consider a dynamical system of "N" particles. If at any time "t" position ~~radius~~ vector of the ith vector of mass "m_i" is "r_i" and having velocity "v_i" then the expression for the K.E denoted by "T" is,

$$T = \frac{1}{2} m_i v_i^2$$

$$= \frac{1}{2} m_i \dot{r}_i^2 \quad \text{——} (1)$$

To express the K.E of the system in terms of generalized coordinates we substitute,

$$\dot{r}_i = \frac{\partial r_i}{\partial q_s} \dot{q}_s + \frac{\partial r_i}{\partial t}$$

$$\text{So, } T = \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial q_s} \dot{q}_s + \frac{\partial r_i}{\partial t} \right) \left(\frac{\partial r_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t} \right)$$

$s = 1, 2, 3, \dots n$

$k = 1, 2, 3 - - n$

$$\Rightarrow T = \frac{1}{2} m_i \left[\frac{\partial r_i}{\partial q_s} \frac{\partial r_i}{\partial q_k} \dot{q}_s \dot{q}_k + \left(\frac{\partial r_i}{\partial t} \right)^2 + 2 \frac{\partial r_i}{\partial q_s} \frac{\partial r_i}{\partial t} \dot{q}_s \right]$$

Put,
$$a_{sk} = m_i \frac{\partial r_i}{\partial q_s} \frac{\partial r_i}{\partial q_k}$$

$$a_s = m_i \frac{\partial r_i}{\partial q_s} \frac{\partial r_i}{\partial t} \quad \& \quad a_o = \left(\frac{\partial r_i}{\partial t} \right)^2 m_i$$

$$\text{So, } T = \frac{1}{2} \left(a_{sk} \dot{q}_s \dot{q}_k + 2 a_s \dot{q}_s + a_o \right)$$

where a_{sk}, a_s, a_o are the functions of q_s and t

Let, $T_2 = \frac{1}{2} a_{sk} \dot{q}_s \dot{q}_k$

$T_1 = \frac{1}{2} a_s \dot{q}_s$ & $T_o = \frac{1}{2} a_o$

$\Rightarrow T = T_2 + 2T_1 + T_0$

We note that T_1 is a homogeneous linear expansion of $\dot{q}_s$ and T_0 is a homogeneous zero degree expansion of velocities ($\dot{q}_s$'s) and the constraints are dependent.

The K.E for stationary constraints is of considerable importance.

In this case we choose the generalized coordinates in such a way that the time "t" does not occur in the expression of cartesian coordinates r_i. Thus the last two terms vanish and the K.E. is,

$$T = \frac{1}{2} a_{sk} \dot{q}_s \dot{q}_k \qquad s,k = 1,2,3 \ldots n$$

$$= \frac{1}{2} a_{sk} (\dot{q}_s)^2$$

Diff. w.r.t $\dot{q}_s$ we get,

$$\frac{\partial T}{\partial \dot{q}_s} = a_{sk} \dot{q}_s$$

Multiplying both sides by $\dot{q}_s$

$$\dot{q}_s \frac{\partial T}{\partial \dot{q}_s} = a_{sk} (\dot{q}_s)^2$$

$$= 2\left[\frac{1}{2} a_{sk} (\dot{q}_s)^2\right]$$

$$\dot{q}_s \frac{\partial T}{\partial \dot{q}_s} = 2T$$

So K.E of the system is quadratic function of generalized velocities.

Show that Lagrange's Eq. of motion are Invariant.

Let us consider a dynamical system of n particles whose configuration at any time "t" is specified by n Lagrangian coordinates $q_1, q_2, \ldots q_n$. Then Lagrange's eq. of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \quad \text{--- (1)}$$

$s = 1, 2, 3, \ldots n$

Now we transform coordinate system from q_s to q_s'

then, $q_s' = q_s'(q_s, t)$

$$\Rightarrow \dot{q}_s' = \frac{\partial q_s'}{\partial q_s}\dot{q}_s + \frac{\partial q_s'}{\partial t}$$

$$\& \quad \frac{\partial \dot{q}_s'}{\partial \dot{q}_s} = \frac{\partial q_s'}{\partial q_s} \quad \text{--- (2)}$$

Also sppose that for new coordinate system i.e for q_s', $L = L'$

So, $L(q_s, \dot{q}_s, t) = L'(q_s', \dot{q}_s', t)$

$$\frac{\partial L}{\partial \dot{q}_s} = \frac{\partial L'}{\partial \dot{q}_s'}\frac{\partial \dot{q}_s'}{\partial \dot{q}_s}$$

$$\frac{\partial L}{\partial \dot{q}_s} = \frac{\partial L'}{\partial \dot{q}_s'}\frac{\partial q_s'}{\partial q_s} \quad \text{--- (3)}$$

using (2)

$$\& \quad \frac{\partial L}{\partial q_s} = \frac{\partial L'}{\partial q_s'}\frac{\partial q_s'}{\partial q_s} + \frac{\partial L'}{\partial \dot{q}_s'}\cdot\frac{\partial \dot{q}_s'}{\partial q_s} \quad \text{--- (4)}$$

Using (3) & (4) in eq (1)

then,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_s'}\frac{\partial q_s'}{\partial q_s}\right) - \left[\frac{\partial L'}{\partial q_s'}\frac{\partial q_s'}{\partial q_s} + \frac{\partial L'}{\partial \dot{q}_s'}\frac{\partial \dot{q}_s'}{\partial q_s}\right] = 0$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_s'}\right)\frac{\partial q_s'}{\partial q_s} + \frac{d}{dt}\left(\frac{\partial q_s'}{\partial q_s}\right)\frac{\partial L'}{\partial \dot{q}_s'} - \frac{\partial L'}{\partial q_s'}\frac{\partial q_s'}{\partial q_s} - \frac{\partial L'}{\partial \dot{q}_s'}\frac{\partial \dot{q}_s'}{\partial q_s} = 0 \quad \text{--- (5)}$$

Consider,

$$\frac{d}{dt}\left(\frac{\partial q_s'}{\partial q_s}\right) = \frac{\partial}{\partial q_k}\left(\frac{\partial q_s'}{\partial q_s}\right)\dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial q_s'}{\partial q_s}\right)$$

$$= \frac{\partial^2 q_s'}{\partial q_k \partial q_s}\dot{q}_k + \frac{\partial^2 q_s'}{\partial t \partial q_s}$$

$$= \frac{\partial}{\partial q_s}\left\{\frac{\partial q_s'}{\partial q_k}\dot{q}_k + \frac{\partial q_s'}{\partial t}\right\}$$

$$= \frac{\partial}{\partial q_s}\dot{q}_s'$$

So eq (5) becomes,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_s'}\right)\frac{\partial q_s'}{\partial q_s} + \frac{\partial \dot{q}_s'}{\partial q_s}\frac{\partial L'}{\partial \dot{q}_s'} - \frac{\partial L'}{\partial q_s'}\frac{\partial q_s'}{\partial q_s} - \frac{\partial L'}{\partial \dot{q}_s'}\frac{\partial \dot{q}_s'}{\partial q_s} = 0$$

$$\left\{\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_s'}\right) - \frac{\partial L'}{\partial q_s'}\right\}\frac{\partial q_s'}{\partial q_s} = 0$$

Since, $q_s' = q_s'(q_s, t)$

$$\Rightarrow \frac{\partial q_s'}{\partial q_s} \neq 0$$

Hence, $\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_s'}\right) - \frac{\partial L'}{\partial q_s'} = 0$

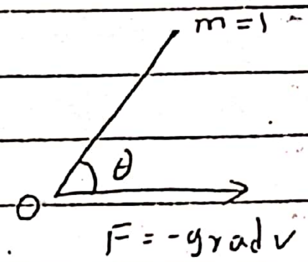
Which shows that Lagrange's Eq. of motion are invariat

Kepler's Problem

A particle of unit mass is moving under the attraction of an inverse square gravitational force to a fixed point "O". Find the Rothean function & Rothean's eq. of motion.

Sol:-

Let us consider a particle of mass "m" is moving in an orbit under the force $-\mu/r^2$. Let at any time "t" the position of particle be $P(x, y)$



then,

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}.$$

as $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$

also, $F = -\frac{\partial V}{\partial r} \Rightarrow \frac{-\mu}{r^2} = -\frac{\partial V}{\partial r}$

$$\Rightarrow \frac{\mu}{r^2} = \frac{\partial V}{\partial r} \Rightarrow V = -\frac{\mu}{r}$$

So, $L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu}{r}$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu}{r} \quad \text{--- (1)}$$

Lagrange's eq. of motion are,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \text{—} \ (2)$$

& $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{—} \ (3)$$

From Lagrangian "L" it appears that "θ" is an ignorable coordinates so,

$$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = \beta \ (\text{constant})$$

$$m r^2 \dot{\theta} = p_\theta \quad \text{or} \quad \dot{\theta} = \frac{p_\theta}{m r^2}$$

As, $L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{\mu}{r}$ See Page 30

$$\Rightarrow L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \frac{p_\theta^2}{m^2 r^4}\right) + \frac{\mu}{r}$$

$$= \frac{1}{2} m \left(\dot{r}^2 + \frac{p_\theta^2}{m^2 r^2}\right) + \frac{\mu}{r}$$

Now, Routhean function is given by,

$$R = L - \dot{\theta} \frac{\partial L}{\partial \dot{\theta}}$$

$$= \frac{1}{2} m \left(\dot{r}^2 + \frac{p_\theta^2}{m^2 r^2}\right) + \frac{\mu}{r} - \dot{\theta} \frac{\partial L}{\partial \dot{\theta}}$$

$$= \frac{1}{2} m \left(\dot{r}^2 + \frac{p_\theta^2}{m^2 r^2}\right) + \frac{\mu}{r} - \frac{p_\theta}{m r^2} (p_\theta)$$

$$= \frac{1}{2} m \left(\dot{r}^2 + \frac{p_\theta^2}{m^2 r^2}\right) + \frac{\mu}{r} - \frac{p_\theta^2}{m r^2}$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{p_\theta^2}{m r^2} + \frac{\mu}{r} - \frac{p_\theta^2}{m r^2}$$

$$= \frac{1}{2} m \dot{r}^2 - \frac{1}{2} \frac{p_\theta^2}{m r^2} + \frac{\mu}{r}$$

$$R = \frac{1}{2} m\dot{r}^2 - \frac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{\mu}{r} \qquad (\because m = 1)$$

Which is the required Routhean function and thus we have seen that system is reduced to a single degreee of freedom.

Now, for Rothean Eq. of motion,

As r is non ignorable coordinate so,

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = 0$$

$$\Rightarrow \quad \frac{d}{dt}(m\dot{r}) - \frac{\partial R}{\partial r} = 0$$

$$\frac{d}{dt}(m\dot{r}) - \left(\frac{p_\theta^2}{mr^3} - \frac{\mu}{r^2}\right) = 0$$

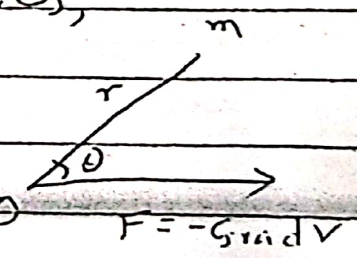
$$m\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{\mu}{r^2} = 0$$

which is the required result.

Q. A particle of mass "m" is attached to a fixed point O by an inverse square force $f = -\mu m/r^2$, where μ is the gravitational co-efficient. Using plane polar coordinates (r, θ) to describe the position of particle, find Hamilton's eq. of motion.

Sol:- In plane polar coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta$$



So, $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Where $F = -\frac{\mu m}{r^2} = -\frac{\partial V}{\partial r}$

$\Rightarrow V = -\frac{\mu m}{r}$

as $L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu m}{r}$

Hamilton's function is,

$$H = p_s \dot{q}_s - L$$

$$= p_r \dot{r} + p_\theta \dot{\theta} - \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu m}{r}$$

where, $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$

& $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$

Thus,

$$H = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{1}{2} m\left(\frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2r^2}\right) - \frac{\mu m}{r}$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{\mu m}{r}$$

Now Hamilton's eq. of motion are,

$\dot{q}_s = \frac{\partial H}{\partial p_s}$ & $\dot{p}_s = -\frac{\partial H}{\partial q_s}$

$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$ $\quad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\mu m}{r^2}$ —(a)

$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$ $\quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$ —(b)

From eq. (b) we get, $p_\theta = \beta$ (constant)

Where β is the constant and eq. (a) reduces to the single second order differential equation,

$$m\ddot{r} - \frac{\beta^2}{mr^3} + \frac{\mu m}{r^2} = 0$$

-: 2nd Poisson's Theorem :-

Let $u(q_i, p_i, t)$ be the first integral of the system, then prove that u is a constant throughout the motion.

Proof:-

Since $u(q_i, p_i, t)$ is the first integral of the system so

$$\left\{ \begin{array}{l} \frac{du}{dt} = 0 \\ \Rightarrow \frac{\partial u}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial u}{\partial p_i}\frac{\partial p_i}{\partial t} + \frac{\partial u}{\partial t} = 0 \\ \frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t} = 0 \end{array} \right.$$

$$[u, H] + \frac{\partial u}{\partial t} = 0 \quad \text{—— (1)}$$

Since the equation of motion in Poisson bracket is,

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

$$\frac{du}{dt} = 0 \qquad \text{(using (1))}$$

$$\Rightarrow \quad u = \text{Constant}$$

Hence u is a constant of motion.

Jacobi's Identity:-

If f, g & h are the functions of q_i, p_i & t then.

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$$

which is called Hamilton Jacobi identity

Proof:- As,

$$[u,v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}.$$

where u & v are the functions of $2n$ variables $q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n$.

Now consider,

$$[f,[g,h]] = \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}[g,h] - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}[g,h]$$

$$= \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial p_j} - \frac{\partial h}{\partial q_j}\frac{\partial g}{\partial p_j}\right)$$

$$- \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial g}{\partial q_j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial q_j}\frac{\partial h}{\partial p_j}\right)$$

$$= \frac{\partial f}{\partial q_i}\left\{\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial p_i \partial p_j} + \frac{\partial h}{\partial p_j}\frac{\partial^2 g}{\partial p_i \partial q_j} - \frac{\partial h}{\partial q_j}\frac{\partial^2 g}{\partial p_i \partial p_j}\right.$$

$$\left. - \frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial p_i \partial q_j}\right\} - \frac{\partial f}{\partial p_i}\left[\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial q_i \partial p_j} + \frac{\partial h}{\partial p_j}\frac{\partial^2 g}{\partial q_i \partial q_j}\right.$$

$$\left. - \frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial q_i \partial p_j} - \frac{\partial h}{\partial p_j}\frac{\partial^2 g}{\partial q_i \partial q_j}\right\}$$

$$= \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial p_i \partial p_j} + \frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_j}\frac{\partial^2 g}{\partial p_i \partial q_j}$$

$$- \frac{\partial f}{\partial q_i}\frac{\partial h}{\partial q_j}\frac{\partial^2 g}{\partial p_i \partial p_j} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\frac{\partial^2 h}{\partial p_i \partial q_j}$$

$$-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial q_i \partial p_j} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial p_j}\frac{\partial^2 g}{\partial q_i \partial q_j}$$

$$+\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_j}\frac{\partial^2 h}{\partial q_i \partial p_j} + \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial p_j}\frac{\partial^2 g}{\partial q_i \partial q_j}$$

$$= 0$$

Similarly

$$[g,[h,f]] = \frac{\partial g}{\partial q_i}\frac{\partial h}{\partial q_j}\frac{\partial^2 f}{\partial p_i \partial p_j} + \frac{\partial g}{\partial q_i}\,\partial!$$

$$[g,[h,f]] = 0$$

& $[h,[f,g]] = 0$

So adding these equations we get

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$$

([redacted]).

-:Hamilton Jacobi Theory:-

Position of a particle in three dimensional space can be specified by three coordinates. If there is no restriction, the three coordinates can be vary independently. However if there are some conditions to be satisfied by the coordinates then they can take values independently. Such conditions can be expressed as relations b/w the coordinates so that the values of the same coordinates can be determined in terms of the values of other coordinates.

The coordinates or parameters which can vary independently and determine the position of the particles are called generalized coordinates.

In case of a system of n particles the complete specification about the location of the particle of the system can be made in terms of $3n$-coordinates. In the absence of any condition (constraints) the $3n$-coordinates can take values independently. If however there are some constraints the coordinates are not free to take any value independently.

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If the number of constraints is m $(<3n)$ then N $(=3n-m)$ coordinates can be given values orbitarly. The other $3n-N$ are determined from the constraints. The maximum number of coordinates or parameters which can be given values independently is unique for a given situation but the coordinates or parameters themselves are not unique. Such coordinates or parameters which can take/ be given values independently are called generalized coordinates.

If $q_1, q_2, \ldots q_N$ are the generalized coordinates then the space coordinates (x_i, y_i, z_i) of the particle of mass m_i $(i=1,2,3,\ldots n)$ can be determined in terms of q_i $(i=1,2,3,\ldots N)$

i.e.
$$x_i = x_i(q_1, q_2, \ldots q_N)$$
$$y_i = y_i(q_1, q_2, \ldots q_N)$$
$$z_i = z_i(q_1, q_2, \ldots q_N) \qquad i=1,2,\ldots n$$

Note that $q_1, q_2, \ldots q_N$ are not the coordinates in real sense of the particles consisting the system. However they are such that the space coordinates (x_i, y_i, z_i) of the particles of mass m_i $(i=1,2,\ldots n)$ can be concerned of $q_1, q_2, \ldots q_N$ are known.

In general the generalized coordinates q_i are the functions of time 't'.

The derivatives of q_i w.r.t. t are called generalized velocities.

If $\dot{q}_i$, $i = 1, 2, 3, \ldots N$ are the generalized velocities then

$$\dot{q}_i = \frac{dq_i}{dt} \qquad i = 1, 2, 3, \ldots N$$

Just like q_i the generalized velocity $\dot{q}_i$ donot represent the velocities of varius particles of the system. However they determine the velocity of the system as

$$\dot{x}_i = \frac{d}{dt}(x_i)$$

$$= \sum_{j=1}^{N} \frac{\partial x_i}{\partial q_j} \frac{dq_j}{dt} = \sum_{j=1}^{N} \frac{\partial x_i}{\partial q_j} \dot{q}_j \text{ etc.}$$

K.E "T" of the system is given by,

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i (\dot{x}_i \dot{x}_i + \dot{y}_i \dot{y}_i + \dot{z}_i \dot{z}_i)$$

$$= \frac{1}{2} \sum_{i=1}^{n} m_i \left[\sum_{j,k=1}^{N} \frac{\partial x_i}{\partial q_j} \dot{q}_j \frac{\partial x_i}{\partial q_k} \dot{q}_k + \sum_{j,k=1}^{N} \frac{\partial y_i}{\partial q_j} \dot{q}_j \frac{\partial y_i}{\partial q_k} \dot{q}_k \right.$$

$$\left. + \sum_{j,k=1}^{N} \frac{\partial z_i}{\partial q_j} \frac{\partial z_i}{\partial q_k} \dot{q}_j \dot{q}_k \right]$$

A system of particles is called a conservative if there exist a function V

of coordinates of the particles such that the forces acting on the particles can be obtained from the relations of the form

$$F_x^i = -\frac{\partial V}{\partial x_i}, \quad F_y^i = -\frac{\partial V}{\partial y_i}, \quad F_z^i = -\frac{\partial V}{\partial z_i}$$

$$i = 1, 2, 3 \ldots n$$

The function V is called the P.E. of the system. The Lagrangian "L" of the system is defined as the difference of K.E & P.E i.e.

$$L = T - V$$

Now V which is a function of the coordinates of particles of the system becomes a function of the generalized coordinates $q_1, q_2, \ldots q_n$ where the expression for space coordinates is substituted by the generalized coordinates. Thus L can be considered as n function of $q_1, q_2 \ldots q_N$ and $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_N$ so,

$$L = L(q_1, q_2, \ldots q_N, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_N)$$

Usually q_i & $\dot{q}_i$ are the function of time "t"

The integral,

$$\int_{t_1}^{t_2} L(q_1, q_2, \ldots q_N, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_N)\, dt$$

is called Hamilton's integral and represent action from t_1 to t_2

Let us consider a dynamical system of N particles whose configuration at any time 't' is specified by n generalized coordinates. Then corresponding Lag

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \quad \text{—(1)}$$

Suppose that out of s particle k particles are ignorable where k < s.
The a.f. k all are present in L

If $\frac{\partial L}{\partial q_i} = 0 \quad i = 1,2,3 \ldots k$

The Lagrange's eq of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - 0 = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_i} = p_i$$

where p_i is the constant of integration called generalized momentum. Lagrang

$$L = L(q_{k+1}, q_{k+2} \ldots q_n; \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n; t)$$

$$L = L(q_j, \dot{q}_s, t)$$

Routhian Define a function of R = r

$$R = L - \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

where $R = R(q_{k+1}, q_{k+2} \ldots q_n, \dot{q}_{k+1}, \dot{q}_{k+2} \ldots \dot{q}_n, p_1, p_2 \ldots p_k, t)$

$$R = R(q_j, \dot{q}_j, p_i, t)$$

$$\delta R = \delta L - \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i$$

$$= \frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s - \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial}{\partial \dot{q}_i} \delta \dot{q}_i$$

$$= \frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j - \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i$$

$$\delta R = \delta L - \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i - \dot{q}_i \delta\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

$$\delta R = \delta L - \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i - \dot{q}_i p_i$$

$$\delta R = \frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s - \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i - \dot{q}_i p_i$$

$$\delta R = \frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j - \dot{q}_i p_i$$

$$\delta R = \frac{\partial R}{\partial q_i}\delta q_j + \frac{\partial R}{\partial \dot{q}_j}\delta \dot{q}_j + \frac{\partial R}{\partial p_i}\delta p_i$$

$$\dot{q}_i = -\frac{\partial R}{\partial p_i}, \quad \frac{\partial R}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_i} \quad \frac{\partial R}{\partial q_i}$$

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The END.

These Notes are Lectures Delivered by.

Sir Saqib. DR.

M.Sc
MATHEMATICS
PAPER 4 OPTION 3

MATHEMATICS

SPECIAL THEORY & RELATIVITY

Definition:- (Relativity)

Theorey of relativity deals with the relation which exist between the physical entities as they appear to different observers who are in motion relative to each other.

Types of Relativity

There are two types of relativity

(i) Special theorey of relativity
It deals with uniform motion.

(ii) General theorey of relativity
It deals with accelerated motion.

Frame of reference

Any set of planes or surfaces by means of which the position of a point

in space is uniquely described.

OR A set of coordinate axes with respect to which measurement are made is called a frame of reference.

There are two types of the frame of reference

(1) Inertial frame of reference

A frame of reference in which Newton's laws valid is called inertial frame of reference.

OR A frame of reference which moves with uniform velocity or at rest is called inertial frame of reference.

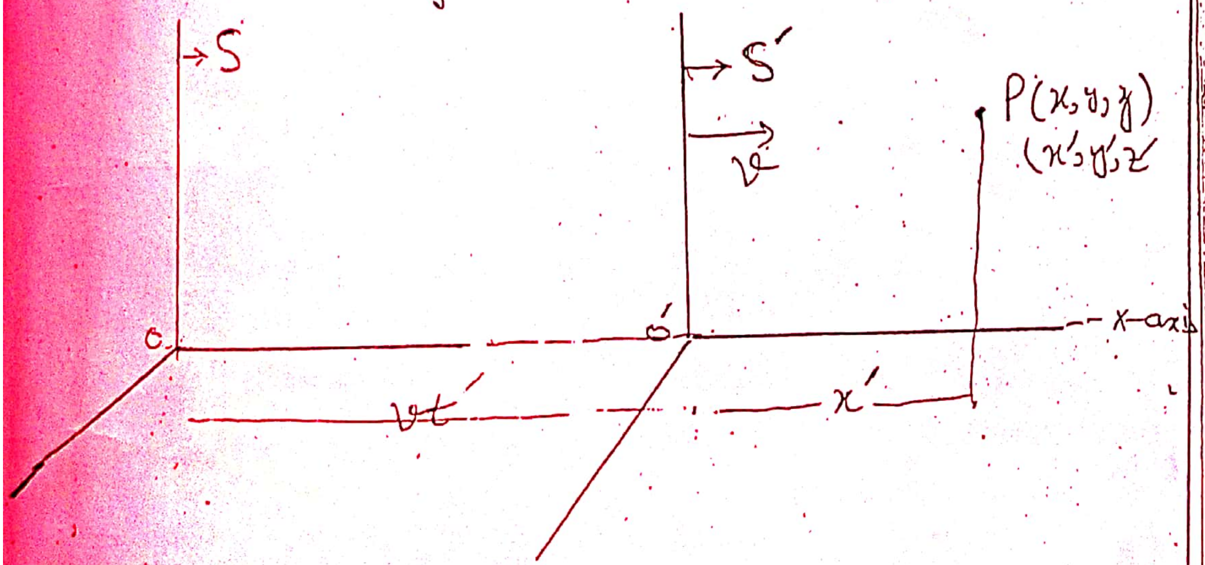
(2) Non inertial frame of reference

A frame of reference in which Newton's laws are

not valid is called non inertial or accelerated frame of reference.

Galallian transformation.

Consider two inertial frames of reference S and 'S′' with origin 'O' and 'O′' respectively. frame S′ is moving with velocity 'V' relative to 'S' along x-axis.



let 'p' be a point whose coordinates ~~are~~ relative to S are (x, y, z) and w.r.t

are (x', y', z').

Suppose both coincides at time $t = 0$. If observer 'O' measure the time 't' and O' measure the time 't'' at some instant. For observer O'

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

which are Galillian transformation.

In matrix form we can write

$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

The corresponding velocities are given by

$$V_x' = V_x - V$$

$$V_y' = V_y$$

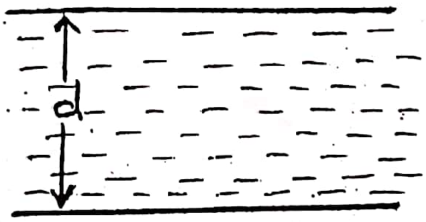
$$V_z' = V_z$$

the inverse Transformation can be obtained by changing v by −v and interchanging primed and unprimed Coordinates.

2001

Michelson and Morely experiment.

let water in a stream is flowing with the uniform velocity $\vec{V}$. let d be the width of the stream. A man whose speed of swimming in still water is $\vec{V}'$, swims a distance 'd' down the stream and up the stream. Time 't' for this two journey is given by



$$t_1 = \frac{d}{V'+V} + \frac{d}{V'-V}$$

$$= \frac{d(v'-v)+d(v'+v)}{(v'+v)(v'-v)}$$

$$= \frac{dv' - dv + dv' + dv}{v'^2 - v^2}$$

(In the second numerator, the two dv terms are crossed out.)

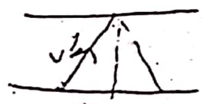
$$t_1 = \frac{2dv'}{v'^2 - v^2} = \frac{\frac{2d}{v'}}{1-\frac{v^2}{v'^2}}$$

Now the man swims right across the streams to the opposite bank and then returns back.

time for this journey is

$$t_2 = \frac{d}{\sqrt{v'^2-v^2}} + \frac{d}{\sqrt{v'^2-v^2}}$$

$$= \frac{2d}{\sqrt{v'^2-v^2}}$$

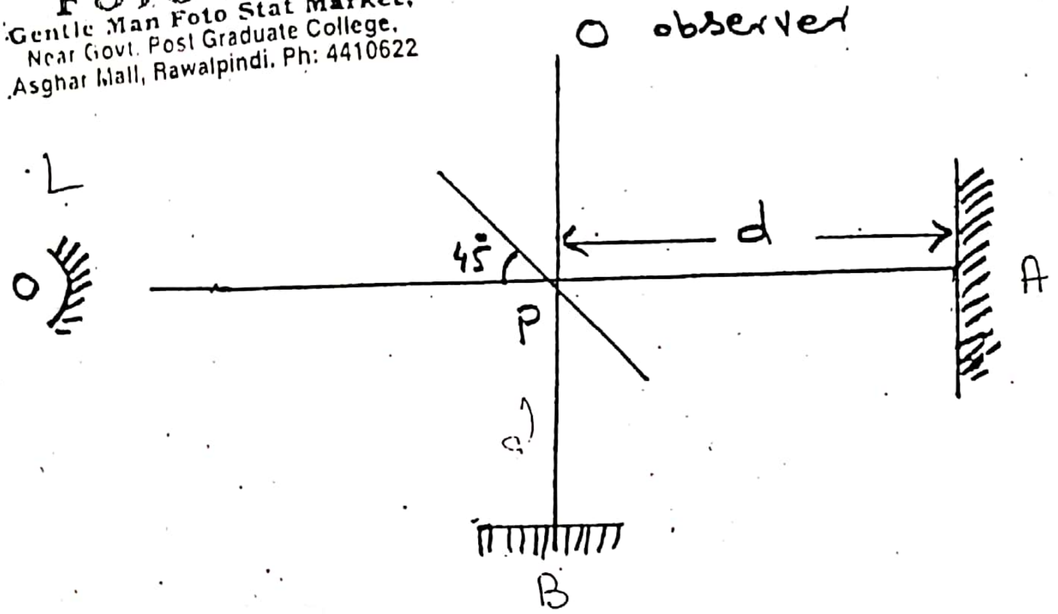


$$t_2 = \frac{2d/v'}{\sqrt{1-\frac{v^2}{v'^2}}}$$

Now $\Delta t = t_1 - t_2$

$$= \frac{2d}{V'}\left[\frac{1}{1-\frac{V^2}{V'^2}} - \frac{1}{\sqrt{1-\frac{V^2}{V'^2}}}\right]$$

$$= \frac{2d}{V'}\left[\frac{1-\sqrt{1-(V/V')^2}}{1-\left(\frac{V}{V'}\right)^2}\right]$$



Now let us consider light is coming from a source 'L' and is partially reflected by a semi silvered glass plate 'P'. The transmitted part is reflected by a mirror A and again reflect at P to reach the observer 'O'. Reflected part

of light from P is again reflected by B and finally reaches at O.

The apparatus is slowly rotated through a right

fig 6 when the direction of arm PA coincides with the direction of the motion of the earth. the time of transitive of two parts of light is given by

$$t_1 = \frac{2dc}{c^2 - v^2}$$

$$t_2 = \frac{2d}{\sqrt{c^2 - v^2}}$$

$$\Delta t = t_1 - t_2 = \frac{2d/c}{1 - \frac{v^2}{c^2}} - \frac{2d/c}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{2d}{c}\left[1 - \left(\frac{v}{c}\right)^2\right]^{-1} - \frac{2d}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$= \frac{2d}{c}\left[1 + \left(\frac{v}{c}\right)^2\right] - \frac{2d}{c}\left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right]$$

Neglecting higher terms

$$= \frac{2d}{c}\left[\frac{1}{2}\left(\frac{v}{c}\right)^2\right]$$

$$\Delta t = \frac{d\,v^2}{c^3}$$

Since 'd' and 'c' are known. It was expected that by calculating Δt it would be possible to find $\vec{v}$ i.e velocity of earth relative to aether. which is the absolute velocity of the earth. Contrary to the experiment failed to give desired result.

Thus earth did not move relative to aether around it. Many explanations were provided but non of them were satisfactory, except for the suggestions of Lorentz and Fitzgerald.

They suggested that due to some unknown dynamical

effects there may be a Contraction of length along the direction of motion. In fact that, for t_1, we get

$$t_1 = \frac{2d'/c}{1 - \frac{v^2}{c^2}}$$

Now for Δt to be zero, $t_1 = t_2$ and hence

$$\frac{2d'/c}{1 - v^2/c^2} = \frac{2d/c}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \quad d' = d\sqrt{1 - v^2/c^2}$$

This is called Lorentz-Fitzgerald Contraction. Lorentz derived this result from his Coordinate transformation.

Einstein postulates of special theorey of relativity.

Einstein stated two pastulate of special theorey of relativity.

(1) Laws of physics have the same form in inertial frame of references.

(2) the velocity of light in empty space is the same for all observers independent of the relative motion of the source and observer.

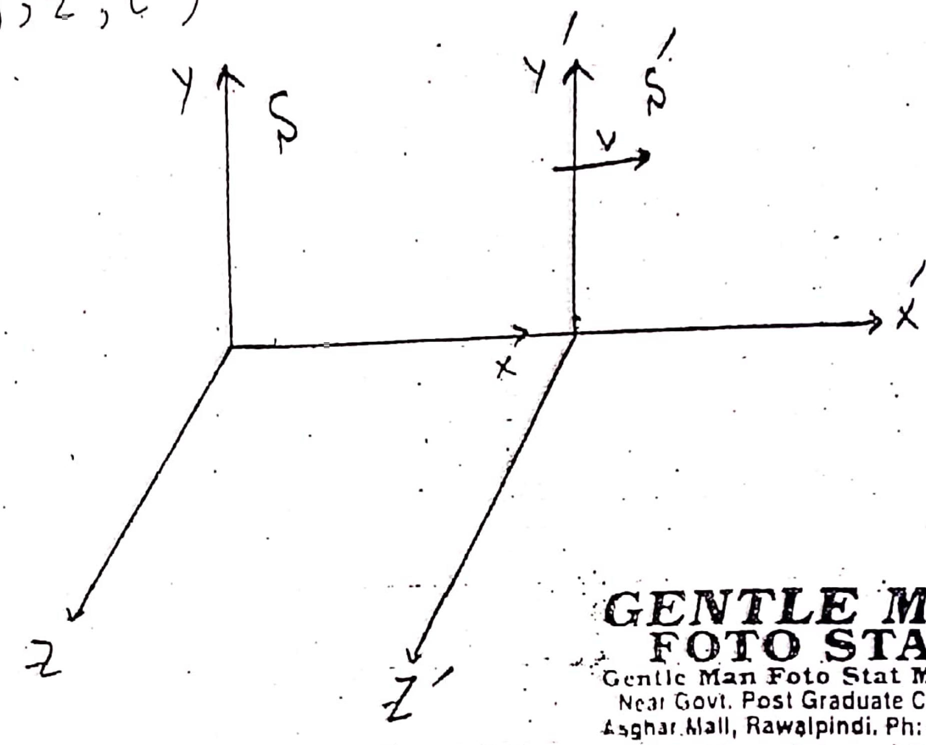
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Lorentz transformation

Consider S and S' be two frames with camman x-axis and S' is moving with uniform velocity 'v' relative to 'S' along the cammon x-axis.

let the coordinate of an event in frame 'S' are (x, y, z, t) and w.r.t S' are

(x', y', z', t').



When $t = t' = 0$

both the frames coincides.

The transformation which relates the coordinates of event in both the frames are related as

$$x' = \alpha(x - vt) \quad \text{——— (1)}$$

$$y' = y \quad \text{——— (2)}$$

$$z' = z \quad \text{——— (3)}$$

$$t' = \beta t + \gamma x \quad \text{——— (4)}$$

At time $t=0$; a spherical wave originated from the origin of S whose equation is given by

$$x^2+y^2+z^2 = c^2t^2 \quad \text{——(5)}$$

In frame S'

$$x'^2+y'^2+z'^2 = c^2t'^2 \quad \text{——(6)}$$

Using (1) → (4) in (6)

$$\alpha^2(x-vt)^2+y^2+z^2 = c^2(\beta t+\gamma x)^2$$

$$\alpha^2x^2+\alpha^2v^2t^2-2\alpha^2vtx+y^2+z^2 = c^2(\beta^2t^2+\gamma^2x^2+2\beta\gamma tx)$$

$$(\alpha^2-c^2\gamma^2)x^2+y^2+z^2-2t(\alpha^2v+\beta\gamma c^2)x = (c^2\beta^2-\alpha^2v^2)t^2 \quad \text{——(7)}$$

Now Comparing the coefficents of x^2, xt and t^2

$$\alpha^2-c^2\gamma^2 = 1 \quad \text{——(8a)}$$

$$\alpha^2v+\beta\gamma c^2 = 0 \quad \text{——(8b)}$$

$$c^2\beta^2-\alpha^2v^2 = c^2 \quad \text{——(8c)}$$

From equation (8b)

$$\alpha^2 = \frac{-c^2\beta\gamma}{v}$$

Put in Equation (8a)

$$-\frac{c^2\beta\gamma}{V} - c^2\gamma^2 = 1$$

$$c^2\beta\gamma + c^2\gamma^2 V = -V$$

$$c^2\gamma(\beta + \gamma V) = -V \quad \text{— (9a)}$$

Putting the value of α^2 in (8c)

$$c^2\beta^2 + \frac{c^2\beta^2\gamma V^2}{V} = c^2$$

$$\beta^2 + \beta^2\gamma V = 1$$

$$\beta(\beta + \gamma V) = 1$$

$$\beta + \gamma V = \frac{1}{\beta} \quad \text{— (9b)}$$

From Equation (9a) and (9b) we have

$$\frac{c^2\gamma}{\beta} = -V$$

$$\Rightarrow \gamma = \frac{-\beta V}{c^2} \quad \text{— (10)}$$

using the value of γ in (8b)

$$\alpha^2 V - \beta^2 V = 0$$

$$\Rightarrow \alpha^2 = \beta^2 \quad \text{— (11)}$$

Equation (11) in (8c)

$$\alpha^2 (c^2 - v^2) = c^2$$

$$\Rightarrow \quad \alpha^2 = \frac{c^2}{c^2 - v^2}$$

$$\alpha = \frac{c}{\sqrt{c^2 - v^2}}$$

$$\Rightarrow \quad \alpha = \frac{1}{\sqrt{1 - v^2/c^2}}$$

which is also the value of β.

$$\Rightarrow \quad \alpha = \beta = \frac{1}{\sqrt{1 - v^2/c^2}}$$

using (10)

$$\gamma = -\frac{1}{\sqrt{1 - v^2/c^2}} \cdot \frac{v}{c^2}$$

$$\Rightarrow \quad \gamma = \frac{-v/c^2}{\sqrt{1 - v^2/c^2}}$$

So actual transformation ~~are~~ is

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad \text{— (A)}$$

$$y' = y \quad \text{— (B)}$$

$$z' = z \quad \text{—— (C)}$$

$$t' = \frac{t}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{—— (D)}$$

these equations (from A to D) are the Lorentz transformation.

Remark:-

(1) if $v << c$

then $\frac{v}{c} = 0$

$$\Rightarrow x' = x - vt$$

$$y' = y \;,\; z' = z$$

$$\& \;\; t' = t$$

which is the classical Galallian transformation.

(2) Inverse of Lorentz transformation can be obtain by changing 'v' to '$-v$' and interchanging primed and unprimed coordinates.

thus inverse Lorentz Transformation are

$$x = \frac{x' + vt}{\sqrt{1 - v^2/c^2}}$$

$$y = y' \quad , \quad z = z'$$

$$t = \frac{t' + \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}}$$

3 If $v = c$ or $v > c$ then denomenator in the Equation (A) and (D) Equations is either zero or Imaginary then Lorentz transformation are meaningless

Facts of Lorentz Transformation on time ~~dil~~ and length

2001 Time dilation :-

Consider a clock rigidly connected with S' frame placed at some point (x'_0, y'_0, z'_0) and moving with

uniform velocity V relative to S along x-axis. then its time is given by

$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1-\frac{v^2}{c^2}}}$$

In S frame the interval $t_2 - t_1$ is related to corresponding S' frame interval $t_2' - t_1'$ by the equation

$$t_2 - t_1 = \frac{t_2' + \frac{v}{c^2}x_0'}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1' + \frac{v}{c^2}x_0'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{t_2' - t_1'}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Thus according to an observer in S frame clock of S' frame appeared to be slow down by the factor $\sqrt{1-\frac{v^2}{c^2}}$.
Now if the clock is fixed in S at point (x_1, y_1, z_1). according to S' Frame

$$t' = \frac{t - \frac{v}{c^2}x_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- The corresponding relation in the interval is

$$t_2' - t_1' = \frac{t_2 - t_1}{\sqrt{1 - v^2/c^2}}$$

i.e to an observer in S' frame the clock appeared to be slow down by the same factor $\sqrt{1 - v^2/c^2}$.

- This is called clocality or Pardox.

Length Contraction:- [2ad]

Let the length of the rigid rod along x-axis be l and l' relative to S and S'. Let the coordinate of end points are $(x_1, 0, 0)$, $(x_2, 0, 0)$ and $(x_1', 0, 0)$ & $(x_2', 0, 0)$ w.r.t frame S and S'

Now $x_1' = \dfrac{x_1 - vt}{\sqrt{1 - v^2/c^2}}$, $x_2' = \dfrac{x_2 - vt}{\sqrt{1 - v^2/c^2}}$

$$l' = x_2' - x_1'$$

$$= \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}$$

$$l' = (x_2 - x_1)\frac{1}{\sqrt{1-v^2/c^2}}$$

∴ Relative to an observer in S' frame rod appears to be contracted by the factor $\sqrt{1-v^2/c^2}$

Relativity of Simultanity

Two events which are simultaneous relative to the frame S but occure at different places are not simultaneous relative to S'

Suppose two events occure at $(x_1, 0, 0)$ and $(x_2, 0, 0)$ and are simultaneous w.r.t frame S

i.e $t_1 = t_2$

Now
$$t_1' = \frac{t_1 - \frac{v}{c^2}x_1}{\sqrt{1-v^2/c^2}}$$

$$t_2' = \frac{t_2 - \frac{v}{c^2}x_2}{\sqrt{1-v^2/c^2}}$$

$$t_2' - t_1' = \frac{\frac{v}{c^2}(x_1 - x_2)}{\sqrt{1-\frac{v^2}{c^2}}} \neq 0$$

$\Rightarrow t_2' \neq t_1'$

$\Rightarrow$ events are not simultanaus

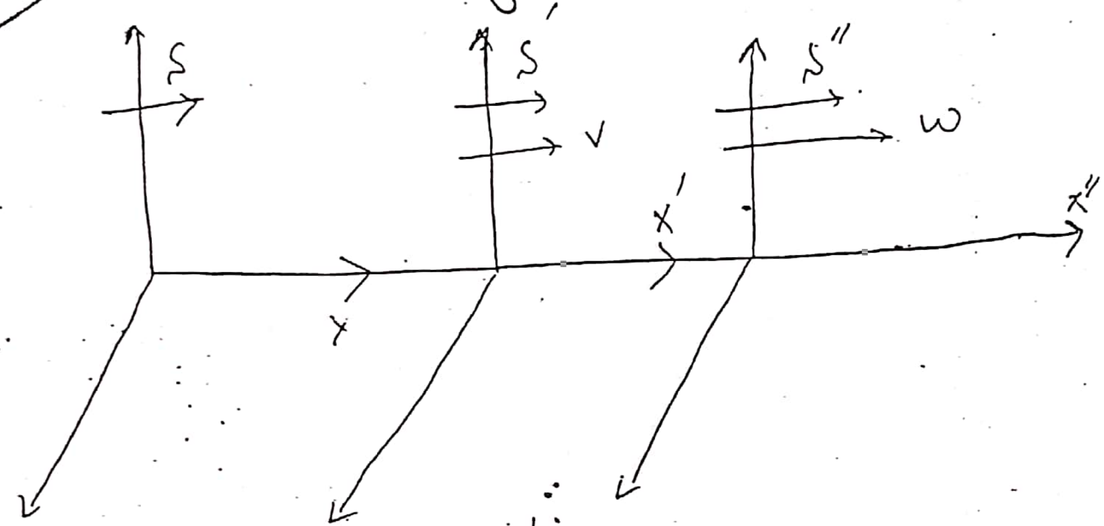
1987

relative to S'.

Relativistic addition of velocities:-

Q Show that the product of two Lorentz Transformation is a Lorentz Transformation.

Sol^n



Suppose a frame S' is moving with velocity v relative to a frame S along common x-axis. Frame S'' is moves relative to S' with velocity w.

Transformation equations from S to S' are

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y \quad , \quad z' = z$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}} \qquad (1)$$

Transformation equation from S' to S'' are

$$\left.\begin{aligned} x'' &= \frac{x' - \omega t'}{\sqrt{1-\omega^2/c^2}} \\ y'' &= y', \quad z'' = z' \\ t'' &= \frac{t' - \frac{\omega}{c^2} x'}{\sqrt{1-\omega^2/c^2}} \end{aligned}\right\} \quad \text{——— (2)}$$

Putting the value of (1) in (2)

$$x'' = \frac{\frac{x - vt}{\sqrt{1-v^2/c^2}} - \frac{\omega\left(t - \frac{v}{c^2}x\right)}{\sqrt{1-v^2/c^2}}}{\sqrt{1-\omega^2/c^2}}$$

$$= \frac{x - vt - \omega t + \frac{\omega v x}{c^2}}{\sqrt{1-\frac{v^2}{c^2}} \cdot \sqrt{1-\frac{\omega^2}{c^2}}}$$

$$= \frac{\left(1 + \frac{\omega v}{c^2}\right)x - (v+\omega)t}{\sqrt{1 - \frac{\omega^2}{c^2} - \frac{v^2}{c^2} + \frac{\omega^2 v^2}{c^4}}}$$

$$= \frac{\left(1 + \frac{\omega v}{c^2}\right)x - (v+\omega)t}{\sqrt{1 + \frac{\omega^2 v^2}{c^4} + 2\frac{\omega v}{c^2} - \frac{\omega^2}{c^2} - \frac{v^2}{c^2} - \frac{2\omega v}{c^2}}}$$

$$x'' = \frac{\left(1+\frac{wv}{c^2}\right)x - (v+w)t}{\sqrt{\left(1+\frac{wv}{c^2}\right)^2 - \left(\frac{v}{c}+\frac{w}{c}\right)^2}}$$

$$= \frac{x - \left(\frac{v+w}{1+\frac{wv}{c^2}}\right)t}{\sqrt{1 - \left(\frac{v+w}{1+\frac{wv}{c^2}}\right)^2 \Big/ c^2}}$$

Let $u = \dfrac{v+w}{1+\frac{vw}{c^2}}$.

$$\therefore \quad x'' = \frac{x-ut}{\sqrt{1-u^2/c^2}}$$

$$y'' = y \,, \quad z'' = z$$

Also $t'' = \dfrac{t' - \frac{w}{c^2}x'}{\sqrt{1-w^2/c^2}}$

$$t'' = \left[\frac{t - \frac{v}{c^2}x}{\sqrt{1-v^2/c^2}} - \frac{w}{c^2}\left(\frac{x-vt}{\sqrt{1-v^2/c^2}}\right)\right] \Big/ \sqrt{1-w^2/c^2}$$

$$= \frac{t - \frac{v}{c^2}x - \frac{wx}{c^2} + \frac{wvt}{c^2}}{\sqrt{1-v^2/c^2}\,\sqrt{1-w^2/c^2}}$$

$$t'' = \frac{t\left(1+\frac{wv}{c^2}\right) - \frac{x}{c^2}(v+w)}{\sqrt{1-\frac{w^2}{c^2}-\frac{v^2}{c^2}+\frac{w^2v^2}{c^4}}}$$

$$= \frac{t\left(1+\frac{wv}{c^2}\right) - \frac{x}{c^2}(v+w)}{\sqrt{\left(1+\frac{vw}{c^2}\right)^2-\left(\frac{v}{c}+\frac{w}{c}\right)^2}}$$

$$= \frac{t - \frac{x}{c^2}\left(\frac{v+w}{1+\frac{wv}{c^2}}\right)}{\sqrt{1-\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)^2 / c^2}}$$

$$= \frac{t - \frac{x}{c^2}u}{\sqrt{1-\frac{u^2}{c^2}}}$$

$\therefore$ S'' moves relative to S with speed $u = \frac{v+w}{1+\frac{wv}{c^2}}$

Special Case

of $u << c$ and $w << c$

then $u = v + w$

Transformation of velocities:-

We find the relativistic transformation law for velocity i.e we would like to see how the velocity in one inertial frame would vary from another inertial frame of reference. Let us consider the motion of a particle with respect to two standered frame of reference S and S'.

If the space time coordinate of the particle in the two frames are (x, y, z, t) and (x', y', z', t') then its velocity in frame S is $u = (u_x, u_y, u_z)$ and $u' = (u_x', u_y', u_z')$ in frame S' are given by

$$u = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \text{ and}$$

$$u' = \left(\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'}\right)$$

By Lorentz transformation, we have

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \quad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

In differential form we obtain

$$dx' = \gamma(dx - v\,dt)$$
$$dy' = dy, \quad dz' = dz$$
$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

As $u'_x = \frac{dx'}{dt'}$

$$u'_x = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - \frac{v}{c^2}dx\right)}$$
$$= \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx}$$
$$= \frac{dt\left(\frac{dx}{dt} - v\right)}{dt\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)}$$
$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x} \quad \left(\because \frac{dx}{dt} = u_x\right)$$

and $u'_y = \frac{dy'}{dt'}$

$$= \frac{dy}{\gamma(dt - \frac{v}{c^2}dx)}$$

$$= \frac{dy\,\frac{1}{\gamma}}{dt\,(1 - \frac{v}{c^2}\frac{dx}{dt})}$$

$$= \frac{\frac{dy}{dt}\cdot\frac{1}{\gamma}}{1 - \frac{v}{c^2}u_x}$$

$$u'_y = \frac{u_y\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c^2}u_x}$$

Also

$$u'_z = \frac{dz'}{dt'}$$

$$u'_z = \frac{dz}{\gamma(dt - \frac{v}{c^2}dx)}$$

$$= \frac{dz\cdot\frac{1}{\gamma}}{dt\,(1 - \frac{v}{c^2}\frac{dx}{dt})}$$

$$u'_z = \frac{u_z\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c^2}u_x}$$

from above

$$U_x' = \frac{U_x - V}{1 - \frac{V}{c^2} U_x}, \quad U_y' = \frac{U_y\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}U_x}$$

$$U_z' = \frac{U_z\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V}{c^2}U_x} \quad \text{--- (1)}$$

These gives the relativistic transformation laws for the components of velocity.

If $V << c$ then V/c can be neglected.

∴ Above equations reduces to

$$U_x' = U_x - V, \quad U_y' = U_y, \quad U_z' = U_z. \quad \text{--- (2)}$$

This is called Newton's or classical's transformation law for the velocity.

2007 General Lorentz Transformation

OR (3-dimensional Lorentz Transformation)

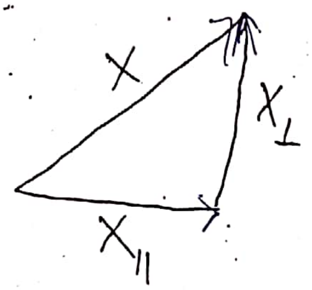
The transformation of space time coordinates in the more general case were the relative velocity of S' w.r.t S is

not parallel to the x - axis and where the rectangular coordinate in S and S′ have arbitrary orientation relative to each other and obviously being obtain by means of a suitable cambination of special rotation of axis in S and S′.

Consider the vector $X = (x, y, z)$ in the frame S and $X' = (x', y', z')$ w.r.t S′. The position vector X can be divided into two components one along the direction of motion i.e $X_{\parallel}$ and the other perpendicular to it. i.e $X_{\perp}$

$$X = X_{\parallel} + X_{\perp} \quad \text{——} (1)$$

There is no effect of uniform motion on $X_{\perp}$



$$\therefore \quad X'_{\perp} = X_{\perp} \quad \text{——} (2)$$

However $X_{\parallel}$ must transform as the X - coordinates transform.

$$X'_{\parallel} = \gamma (X_{\parallel} - vt) \quad \text{--- (3)}$$

where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

Now $X \cdot V = (X_{\parallel} + X_{\perp}) \cdot V$

$$= X_{\parallel} \cdot V + X_{\perp} \cdot V$$

$$X \cdot V = X_{\parallel} V \quad (\because X_{\perp} \cdot V = 0$$

$$\Rightarrow X_{\parallel} = \left(\frac{X \cdot V}{V \cdot V}\right) \cdot V \quad \text{--- (4)}$$

Now from equation (1)

$$X_{\perp} = X - X_{\parallel}$$

$$X_{\perp} = X - \left(\frac{X \cdot V}{V \cdot V}\right) V \quad \text{--- (5)}$$

Adding (2) and (3)

$$X'_{\parallel} + X'_{\perp} = \gamma (X_{\parallel} - vt) + X_{\perp}$$

$$X' = \gamma (X_{\parallel} - vt) + X - \left(\frac{X \cdot V}{V \cdot V}\right) V$$

$$= X + \gamma \left(\left(\frac{X \cdot V}{V \cdot V}\right) V - vt \right) - \left(\frac{X \cdot V}{V \cdot V}\right) V$$

$$X' = X + V\left[\left(\frac{X\cdot V}{V\cdot V}\right)(\gamma-1) - \gamma t\right] \quad \text{— (A)}$$

and $$t' = \gamma\left[t - \frac{V\cdot x}{c^2}\right] \quad \text{— (B)}$$

Equation (A) and equation (B) are the most general Lorentz transformation. The time dilation formula is clearly not changed by the generalization.

Length contraction formula in most general Lorentz transformation

for this purpose

$$t_1' = t_2'$$

$$\Rightarrow \gamma\left(t_1 - \frac{v\cdot x_1}{c^2}\right) = \gamma\left(t_2 - \frac{v\cdot x_2}{c^2}\right)$$

$$t_1 - t_2 = \frac{v\cdot x_1}{c^2} - \frac{v\cdot (x_2)}{c^2}$$

Now if we define

$\underline{l} = \underline{x}_1 - \underline{x}_2$ and $\underline{l}' = \underline{x}_1' - \underline{x}_2'$

$$\underline{l}' = \underline{x}_1 + \underline{v}\left[\frac{\underline{x}_1\cdot \underline{v}}{v\cdot v}(\gamma-1) - \gamma t_1\right]$$

$$- \underline{x}_2 - \underline{v}\left[\underline{x}_2\cdot\underline{v}(\gamma-1) - \gamma t_2\right]$$

$$\underline{r}' = (\underline{x}_1 - \underline{x}_2) + \underline{v}\left[(\underline{x}_1 - \underline{x}_2)\cdot\frac{\underline{v}(\gamma - 1)}{\underline{v}\cdot\underline{v}} - \gamma(t_1 - t_2)\right]$$

$$\underline{r}' = \underline{r} + \underline{v}\left[\frac{\underline{r}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}(\gamma - 1) - \gamma\frac{\underline{v}\cdot(\underline{x}_1 - \underline{x}_2)}{c^2}\right]$$

$$= \underline{r} + \underline{v}\left[\frac{\underline{r}\cdot\underline{v}(\gamma-1)}{\underline{v}\cdot\underline{v}} - \gamma\frac{\underline{v}\cdot\underline{r}}{c^2}\right]$$

$$= \underline{r} + \underline{v}\,\frac{\underline{v}\cdot\underline{r}}{\underline{v}\cdot\underline{v}}\left[\gamma - 1 - \frac{\gamma}{c^2}(\underline{v}\cdot\underline{v})\right]$$

$$\underline{r} + \underline{v}\,\frac{\underline{v}\cdot\underline{r}}{\underline{v}\cdot\underline{v}}\left[\gamma\left(1 - \frac{v^2}{c^2}\right) - 1\right]$$

$$= \underline{r} + \underline{v}\,\frac{\underline{v}\cdot\underline{r}}{\underline{v}\cdot\underline{v}}\left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(1 - \frac{v^2}{c^2}\right) - 1\right]$$

$$= \underline{r} + \underline{v}\,\frac{\underline{v}\cdot\underline{r}}{\underline{v}\cdot\underline{v}}\left[\sqrt{1 - \frac{v^2}{c^2}} - 1\right]$$

$$\underline{r}' = \underline{r} + \underline{v}\,\frac{\underline{v}\cdot\underline{r}}{\underline{v}\cdot\underline{v}}\left(\gamma^{-1} - 1\right)$$

which is the length contraction formula.

General velocity addition formula

To derive the general velocity addition formula we proceed as follows

The general Lorentz transformation are given by

$$x' = x + v\left[\frac{x \cdot v}{v \cdot v}(\gamma - 1) - \gamma t\right]$$

$$t' = \gamma\left(t - \frac{v \cdot x}{c^2}\right)$$

In differential form

$$\Rightarrow dx' = dx + v\left[\frac{dx \cdot v}{v \cdot v}(\gamma - 1) - \gamma dt\right]$$

&

$$dt' = \gamma\left(dt - \frac{v \cdot dx}{c^2}\right)$$

Now

$$u' = \frac{dx'}{dt'}$$

$$u' = \frac{dx + v\left[\frac{dx \cdot v}{v \cdot v}(\gamma - 1) - \gamma dt\right]}{\gamma\left(dt - \frac{v \cdot dx}{c^2}\right)}$$

$\div$ing numerator and denominator we get

$$u' = \frac{\frac{dx}{dt} + v\left[\frac{\frac{dx}{dt} \cdot v(\gamma - 1)}{v \cdot v} - \gamma\right]}{\gamma\left(1 - \frac{v \cdot \frac{dx}{dt}}{c^2}\right)}$$

$$u' = \frac{u + v\left[\frac{u \cdot v(\gamma - 1)}{v \cdot v} - \gamma\right]}{\gamma\left(1 - \frac{v \cdot u}{c^2}\right)}$$

which is the velocity addition formula.

Minkowski coordinates and Minkowski space:-

Consider the two inertial frames S and S' with origin at O and O' respectively. Frame S' is in motion relative to S.

Also $t = 0 = t'$.

both the frames coincides

Further consider at $t=0$ a spherical wave is originated from the origin then its equation in frame S is

$$x^2 + y^2 + z^2 = c^2t^2$$

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \quad \text{——— (1)}$$

and $x'^2 + y'^2 + z'^2 = c^2t'^2$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \quad \text{——— (2)}$$

Since the lorentz transformation is an orthogonal transformation.

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

$$x^2 + y^2 + z^2 + (ict)^2 = x'^2 + y'^2 + z'^2 + (ict')^2$$

$$\Rightarrow \sum_{i=1}^{4} x_i^2 = \sum_{i=1}^{4} x_i'^2$$

where

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict$$

$$x_1' = x', \quad x_2' = y', \quad x_3' = z', \quad x_4' = ict'$$

The coordinates x_1, x_2, x_3, x_4 are called Minkowski coordinates and the space determined by them is called Minkowski space.

1999 Lorentz transformation in term of Minkowski coordinate.

OR Lorentz transformation in 4-dimension:

Lorentz transformation can be written as

$$\left.\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad y' = y \\ z' &= z \\ t' &= \frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}} \end{aligned}\right\} \quad \text{--- (1)}$$

In term of Minkowski coordinates

$x = x_1$ $\quad$ $x' = x_1'$

$y = x_2$ $\quad$ $y' = x_2'$

$z = x_3$ $\quad$ $z' = x_3'$

$ict = x_4$ $\quad$ $ict' = x_4'$

$\Rightarrow t = \frac{x_4}{ic}$ $\quad$ $\Rightarrow t' = \frac{x_4'}{ic}$

(1) $\Rightarrow$

$$x_1' = \frac{x_1 - V\left(\frac{x_4}{ic}\right)}{\sqrt{1 - V^2/c^2}}$$

$$x_2' = x_2 \;,\; x_3' = x_3$$

$$\frac{x_4'}{ic} = \frac{\frac{x_4}{ic} - \frac{V}{c^2} x_1}{\sqrt{1 - V^2/c^2}}$$

Now $x_1' = \frac{x_1}{\sqrt{1 - V^2/c^2}} + \frac{i\frac{V}{c} x_4}{\sqrt{1 - V^2/c^2}}$

Similarly $x_4' = \frac{x_4}{\sqrt{1 - V^2/c^2}} - \frac{i\frac{V}{c} x_1}{\sqrt{1 - V^2/c^2}}$

$$\Rightarrow x_4' = \frac{-\frac{iv x_1}{c}}{\sqrt{1-v^2/c^2}} + \frac{x_4}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow x_1' = \cos\phi \, x_1 + \sin\phi \, x_4$$

$$x_4' = -\sin\phi \, x_1 + \cos\phi \, x_4$$

where $\cos\phi = \frac{1}{\sqrt{1-v^2/c^2}}$, $\sin\phi = \frac{iv/c}{\sqrt{1-v^2/c^2}}$

$\Rightarrow$ Lorentz transformation is a rotation through an angle ϕ in $X_1 X_4$ - Plane.

Lorentz transformation are

$$x_1' = \cos\phi \, x_1 + 0x_2 + 0x_3 + \sin\phi \, x_4$$

$$x_2' = 0x_1 + x_2 + 0x_3 + 0x_4$$

$$x_3' = 0x_1 + 0x_2 + x_3 + 0x_4$$

$$x_4' = -\sin\phi \, x_1 + 0x_2 + 0x_3 + \cos\phi \, x_4$$

~~In matrix form~~

Matrix of this transformation is

$$\begin{bmatrix} \cos\phi & 0 & 0 & \sin\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\phi & 0 & 0 & \cos\phi \end{bmatrix}$$

Inverse Transformation Can be obtained by inverse matrix.

Proper time

Consider a Particle is moving with the velocity v in an inertial frame of reference.

Let (x, y, z) and $(x+dx, y+dy, z+dz)$ be the two Position of the Particle at time 't' and $t+dt$ respectively which is noted by a clock fixed in that frame. These two Positions Can be considered as two events represented by

the points (x, y, z, t) and $(x+dx, y+dy, z+dz, (t+dt))$ in a four dimensional space. The interval ds between the events is given by

$$(ds)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

Since $(ds)^2$ is an invariant quantity. ∴ on the frame moving with the particle equation ① becomes

$$(ds)^2 = c^2(d\tau)^2 \quad \text{—— ②}$$

Since relative to S'

$$dx = 0, \quad dy = 0, \quad dz = 0$$

where $d\tau$ is the time noted by a clock fixed in the frame moving with particle.

Since ds is invariant

From ① and ②

$$c^2(d\tau)^2 = c^2(dt)^2 - (dr)^2$$

$$(d\tau)^2 = (dt)^2 - \frac{1}{c^2}(dr)^2$$

$$(d\tau)^2 = \left[1 - \left(\frac{dr}{dt}\right)^2 \cdot \frac{1}{c^2}\right] (dt)^2$$

$$d\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt$$

where $v = \frac{dr}{dt}$

This is called proper time interval.

or $$\tau = \int_{\tau_1}^{\tau_2} \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt$$

Explanation of interval in space and time

In the theorey of relativity distance and time are relative entities. However a quantity called interval which is same for all observers can be drived from distance and time.

In special theorey of relativity interval is obtained as follow:

Take the square of the distance b/w the events and square of the distance travelled by light in time b/w the events. Subtract lesser of these from the greater and result is defined as the square of the interval b/w the events. The interval is same for all observer represents a geniune physical relation b/w the events while the time and distance donot.

The intervals are classified as fallow

Time like interval:-

the interval is time like when the time b/w the events is larger than light would take to travel from place of ~~the~~ one event to the place of ather event i.e

$$c^2 > v^2$$

Light like interval :-

The interval is light like when the time b/w the events is exactly equal to the time taken by light to travel from the place of one event to the place of other event. i.e $c^2 = v^2$

Space like ~~ev~~ interval :-

The interval is space like when the time b/w the events is less then light to travel from place of one event to the place of other event i.e $c^2 < v^2$

V.V Imp — 1987, 1988

Four velocity vectors

let S be an inertial frame and let P(x, y, z) be in motion relative to S then

$$d\tau = \left(1 - \frac{V^2}{c^2}\right)^{1/2} dt \quad \text{—— (1)}$$

recorded by a clock moving with particle between two events (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) will be given by

$$\tau_1 - \tau_2 = \int_{\tau_1}^{\tau_2} \left(1 - \frac{V^2}{c^2}\right)^{1/2} dt$$

If x_1, x_2, x_3 and x_4 are Minkowski Coordinates of the event then we have

$$\dot{x}_1 = V_x, \quad \dot{x}_2 = V_y$$

$$\dot{x}_3 = V_z, \quad \dot{x}_4 = ic$$

Now $$\frac{dx_i}{d\tau} = \frac{dx_i}{dt} \cdot \frac{dt}{d\tau} \quad \text{—— (2)}$$

$$\frac{dx_i}{d\tau} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \frac{dx_i}{dt} \quad \text{—— (3)}$$

We therefore define the four velocity V_μ by the equation

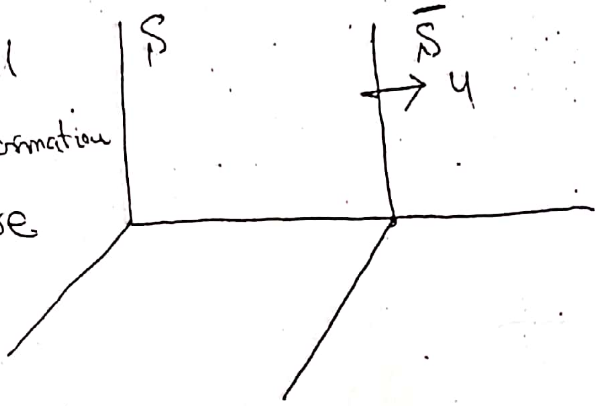
$$\bar{V}_\mu = \left(\frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau}; \frac{dx_4}{d\tau}\right)$$

$$\bar{V}_\mu = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(V_x, V_y, V_z, ic\right) \quad (4)$$

$\bar{V}_\mu$ is called four velocity vector.

Transformation equation for component of 4-velocity vector

Consider a frame $\bar{S}$ is moving with respect to another frame S along common x-axis with velocity u



The Lorentz transformation from S to $\bar{S}$ are

$$\bar{x} = \frac{x-ut}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\bar{y} = y, \quad \bar{z} = z$$

$$\bar{t} = \frac{t-\frac{u}{c^2}x}{\sqrt{1-\frac{u^2}{c^2}}}$$

Now

$$\frac{d\bar{x}}{d\bar{t}} = \frac{d}{dt}\left(\frac{x-ut}{\sqrt{1-\frac{u^2}{c^2}}}\right)\frac{dt}{d\bar{t}} \quad (5)$$

$$= \frac{\frac{dx}{dt} - u}{\sqrt{1-u^2/c^2}} \frac{dt}{d\bar{t}}$$

$$= \frac{V_x - u}{\sqrt{1-u^2/c^2}} \frac{dt}{d\bar{t}}$$

As $\bar{t} = \dfrac{t - \frac{u}{c^2}x}{\sqrt{1-u^2/c^2}}$

$$\frac{d\bar{t}}{dt} = \frac{1 - \frac{u}{c^2}V_x}{\sqrt{1-u^2/c^2}}$$

Thus -

$$\frac{d\bar{x}}{d\bar{t}} = \frac{V_x - u}{\sqrt{1-u^2/c^2}} \cdot \frac{\sqrt{1-u^2/c^2}}{1 - \frac{u}{c^2}V_x}$$

$$\frac{d\bar{x}}{d\bar{t}} = \frac{V_x - U}{1 - \frac{u}{c^2}V_x}$$

$$\Rightarrow \bar{V}_x = \frac{V_x - u}{1 - \frac{u}{c^2}V_x}$$

Similarly

$$\bar{V}_y = \frac{V_y\sqrt{1-u^2/c^2}}{1 - uV_x/c^2}$$

$$\& \quad \bar{V}_z = \frac{V_z \sqrt{1-u^2/c^2}}{1-\frac{uV_x}{c^2}} \quad \text{--- (6)}$$

Equation (6) are transformation equation for component of 4 - velocity vectors.

Remark If u and V are small as compared to c then

$$\bar{V}_x = V_x - u$$

$$\bar{V}_y = V_y , \quad \bar{V}_z = V_z$$

(2) If a light wave of pulse travel along the x-axis with velocity c relative to frame S

$$\Rightarrow V_x = c , \quad V_y = V_z = 0$$

$\therefore$ Relative to S′ frame

$$\bar{V}_x = \frac{c-u}{1-\frac{uc}{c^2}} = \frac{c-u}{\frac{c-u}{c}}$$

$$\bar{V}_x = c$$

Similarly $\bar{V}_y = 0$, $\bar{V}_z = 0$
i.e Velocity of light relative to frame S' is also c along x- axis.

Transformation Equation for Component of four acceleration vectors.

Since transformation equation for four velocity vectors are

$$\bar{V}_x = \frac{V_x - u}{1 - \frac{u V_x}{c^2}}$$

$$\bar{V}_y = \frac{\sqrt{1 - \frac{u^2}{c^2}}\, V_y}{1 - \frac{u V_x}{c^2}}$$

$$\bar{V}_z = \frac{\sqrt{1 - u^2/c^2}\, V_z}{1 - \frac{u V_x}{c^2}}$$

Now

$$\frac{d\bar{V}_x}{d\bar{t}} = \frac{d\bar{V}_x}{dt} \cdot \frac{dt}{d\bar{t}} \quad \text{--- (1)}$$

As

$$\bar{t} = \frac{t - \frac{u}{c^2}x}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\frac{d\bar{t}}{dt} = \frac{1 - \frac{u}{c^2}V_x}{\sqrt{1-\frac{u^2}{c^2}}} \Rightarrow \frac{dt}{d\bar{t}} = \frac{\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{u}{c^2}V_x}$$

(1) $\Longrightarrow$

$$\bar{a}_x = \frac{d}{dt}\left(\frac{V_x - u}{1-\frac{uV_x}{c^2}}\right) \cdot \frac{\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{uV_x}{c^2}}$$

$$= \frac{\left(1-\frac{uV_x}{c^2}\right)a_x - (V_x-u)\left(-\frac{ua_x}{c^2}\right)}{\left(1-\frac{uV_x}{c^2}\right)^2} \cdot \frac{\sqrt{1-\frac{u^2}{c^2}}}{\left(1-\frac{uV_x}{c^2}\right)}$$

$$= \frac{\left(a_x - \frac{uV_xa_x}{c^2} + \frac{uV_xa_x}{c^2} - \frac{u^2a_x}{c^2}\right)\sqrt{1-\frac{u^2}{c^2}}}{\left(1-\frac{uV_x}{c^2}\right)^3}$$

$$= \frac{a_x\left(1-\frac{u^2}{c^2}\right)\sqrt{1-\frac{u^2}{c^2}}}{\left(1-\frac{uV_x}{c^2}\right)^3}$$

$$\bar{a}_x = \frac{\left(1-\frac{u^2}{c^2}\right)^{3/2}a_x}{\left(1-\frac{uV_x}{c^2}\right)^3} \quad \text{— (2)}$$

$$\frac{d\bar{V}_y}{d\bar{t}} = \frac{d\bar{V}_y}{dt} \cdot \frac{dt}{d\bar{t}}$$

$$\bar{a}_y = \frac{d}{dt}\left(\frac{\sqrt{1-u^2/c^2}\, V_y}{1-\frac{uV_x}{c^2}}\right) \cdot \frac{\sqrt{1-u^2/c^2}}{1-u/c^2\, V_x}$$

$$= \left[\frac{(1-uV_x/c^2)\,a_y - V_y\left(-\frac{u a_x}{c^2}\right)}{\left(1-\frac{uV_x}{c^2}\right)^2}\right] \cdot \frac{1-u^2/c^2}{1-u/c^2\, V_x}$$

$$= \frac{a_y - \frac{uV_x a_y}{c^2} + \frac{uV_y a_x}{c^2}}{\left(1-\frac{uV_x}{c^2}\right)^2}$$

$$\bar{a}_y = \left[a_y + \frac{u/c^2\, V_y a_x}{\left(1-\frac{uV_x}{c^2}\right)}\right]\frac{(1-u^2/c^2)}{(1-u/c^2\, V_x)^2} \quad (3)$$

Similarly,

$$\bar{a}_z = \left(a_z + \frac{u/c^2\, V_z a_x}{1-\frac{uV_x}{c^2}}\right)\frac{(1-u^2/c^2)}{\left(1-\frac{uV_x}{c^2}\right)^2} \quad (4)$$

Mass and Momentum

Since time and space are relative it is not reasonable to belive that the Conservative laws of classical mechanics are also valid in relativistic mechanics. However we suppose there are certain invariant quantities associated with colliding particles such that

$$M_1U_1 + M_2U_2 = M_1V_1 + M_2V_2 \quad \text{--- (1)}$$

Where U's and V's are the four velocity vector before and after collision.

Equation (1) may be stated as

$$\sum M \bar{V}_\mu = \text{Constant} \quad \text{--- (2)}$$

$$\sum \left(1 - \frac{V^2}{c^2}\right)^{-1/2} M\,(V, ic) = \text{Constant} \quad \text{--- (3)}$$

$$\Rightarrow \sum m(V, ic) = \text{Constant} \quad \text{--- (4)}$$

where

$$m = \left(1-\frac{v^2}{c^2}\right)^{-1/2} \mu_1 = \frac{M}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{--- (5)}$$

Equation (4) $\Rightarrow$

$$\left.\begin{aligned} \sum m V_\mu &= \text{Constant} \\ \sum m &= \text{Constant} \end{aligned}\right\} \quad \text{--- (6)}$$

M is called rest mass of the particle and is denoted by m_o

Equation (5) becomes

$$m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{--- (7)}$$

when $v < c$ then

$$m = m_o$$

It is reasonable to call m as the mass of the particle at instant 't'.

$$m = \frac{M}{\sqrt{1-v^2/c^2}}$$

Four momentum Vectors

The four momentum vector of a particle is defined by $P_\mu = m_o V_\mu$ —— (8)

where m_o is the rest mass of the particle and 'V_μ' is the four velocity vectors.

$$\vec{P}_\mu = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}(v, ic) \quad \text{where} \quad V_\mu = \frac{(v, ic)}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= m(v, ic)$$

$$= (mv, imc)$$

$$\vec{P}_\mu = (P, imc) \quad \text{—— (9)}$$

$$= (P_x, P_y, P_z, imc)$$

$$P_\mu = (P_1, P_2, P_3, P_4)$$

where $P_4 = imc$.

Clearly transformation equation for the component of $\vec{P}_\mu$ from S to S′ frame are

$$\left.\begin{aligned} \bar{P}_1 &= P_1 \cos\alpha + P_4 \sin\alpha \\ \bar{P}_2 &= P_2 \\ \bar{P}_3 &= P_3 \\ \bar{P}_4 &= -P_1 \sin\alpha + P_4 \cos\alpha \end{aligned}\right\} \quad (10)$$

These transformation equation can be obtained by multiplying m_0 through out with

$$\bar{V}_1 = V_1 \cos\alpha + V_4 \sin\alpha$$

$$\bar{V}_2 = V_2$$

$$\bar{V}_3 = V_3$$

$$\bar{V}_4 = -V_1 \sin\alpha + V_4 \cos\alpha$$

Equation (10) is equivalent to

$$\left.\begin{aligned} \bar{P}_x &= \frac{P_x - mu}{\sqrt{1 - v^2/c^2}} \\ \bar{P}_y &= P_y, \quad \bar{P}_z = P_z \end{aligned}\right\} \quad (11)$$

$$\bar{m} = \frac{m - mu P_x}{\sqrt{1 - u^2/c^2}} \quad (12)$$

Equation (12) shows that mass of the particle depends upon the motion of the particle and the observer.

Q Prove that $p^2 - E^2/c^2$ is an invariant quantity equal to $-m_o^2c^2$

i.e $p^2 - E^2/c^2 = -m_o^2 c^2$

under the Lorentz transformation. Also derive energy momentum equation

i.e $E^2 = p^2c^2 + m_o^2c^4$

where p is the classical momentum and m_o is the rest mass.

Proof

As we know that

$$P_\mu = (p, imc)$$

$$= \left(p, \frac{imc^2}{c}\right)$$

$$P_\mu = \left(p, \frac{iE}{c}\right)$$

$$P_\mu \cdot P_\mu = \left(p, \frac{iE}{c}\right) \cdot \left(p, \frac{iE}{c}\right)$$

$$= p^2 - \frac{E^2}{c^2}$$

$$= m^2v^2 - \frac{1}{c^2}(mc^2)^2$$

$$= m^2v^2 - m^2c^2$$

$$= m^2(v^2 - c^2)$$

$$= \frac{m_o^2}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2}(v^2 - c^2)$$

$$= -\frac{m_o^2c^2(v^2-c^2)}{(v^2-c^2)}$$

$$= -m_o^2c^2$$

Hence

$$p^2 - E^2/c^2 = -m_o^2c^2 \quad \text{--- (1)}$$

$\Rightarrow$ $p^2 - E^2/c^2$ is an invariant quantity since m_o and c are constant.

(1) $\Rightarrow$

$$p^2 - E^2/c^2 = -m_o^2c^2$$

$$\Rightarrow \quad p^2c^2 - E^2 = -m_o^2c^4$$

$$\Rightarrow \quad E^2 = p^2c^2 + m_o^2c^4$$

which is called energy momentum equation

1996 **Four Force vector**

In special theory of relativity we are free for any definition of force which reduces to the classical expression for classical situation.

From a mathematical point of view it is convenient to introduce the concept of force by defining a 4 vector 'F_μ' called four force vector as the rate of change of momentum w.r.t time

$$F_\mu = \frac{d}{d\tau}(P_\mu) \quad \text{——} (1)$$

We check whether space component of F_μ reduced to classical situation.

We first write down the space component of F_μ we have

$$F_1 = \frac{dP_1}{d\tau}$$

$$F_1 = \frac{dP_1}{dt} \cdot \frac{dt}{d\tau}$$

$$= \gamma(v) \cdot \frac{dP_1}{dt}$$

$$= \gamma(v) \frac{d}{dt} (m V_x)$$

$$F_1 = m_0 \gamma(v) \frac{d}{dt} (\gamma(v) V_x) \quad \text{— (2a)}$$

Similarly

$$F_2 = m_0 \gamma(v) \frac{d}{dt} (\gamma(v) V_y) \quad \text{— (2b)}$$

$$F_3 = m_0 \gamma(v) \frac{d}{dt} (\gamma(v) V_z) \quad \text{— (2c)}$$

We note that when v is very very small then c then special components F_1, F_2, F_3 reduces to classical expression for the component of force

Further both sides of equation ① transform as 4-vector. So equation (2a), (2b) and (2C) can be taken as expressing the correct laws of motion in relativistic mechanics.

The 4th component of F_μ is given by

$$F_4 = \frac{dP_4}{d\tau}$$

$$= \frac{dP_4}{dt} \cdot \frac{dt}{d\tau}$$

$$= \gamma(v) \cdot \frac{d}{dt}(imc)$$

$$= ic\gamma(v)\frac{dm}{dt} \quad \text{—— (2d)}$$

When we are considering a system of particles in special theory of relativity a difficulty arises in using equation ①.

Since we have to use different

Proper time for each Particle.

A different definition of force which is more Convenient when we are Considering a system of particle is given by

$$f_\mu = \frac{d}{dt}(P_\mu) \quad \text{———} (3)$$

From above we have

$$f_1 = \frac{d}{dt}(mV_x)$$

where m is the mass of the particle when it is moving with velocity v.

In general

$$f = \frac{d}{dt}(mV)$$

Moreover $f_4 = \frac{d}{dt}(P_4) = \frac{d}{dt}(imc)$

$$= i\dot{m}c$$

Thus $f_4 = (f_1, f_2, f_3, i\dot{m}c)$

$$= (f, i\dot{m}c)$$

Relation between F and f

$$F = \frac{d}{dt}(mv)$$

$$= \frac{d}{d\tau}(P, imc)$$

$$= \frac{d}{dt}(P, imc)\frac{dt}{d\tau}$$

$$= \left(1-\frac{v^2}{c^2}\right)^{-1/2}\frac{d}{dt}(P, imc)$$

$$\left(1-\frac{v^2}{c^2}\right)^{-1/2}(\dot{P}, i\dot{m}c)$$

$$= \left(1-\frac{v^2}{c^2}\right)^{-1/2}(\dot{P}, i\dot{m}c)$$

$$\Rightarrow \quad F_1 = \left(1-\frac{v^2}{c^2}\right)^{-1/2} f_x$$

$$F_2 = \left(1-\frac{v^2}{c^2}\right)^{-1/2} f_2$$

$$F_3 = \left(1-\frac{v^2}{c^2}\right)^{-1/2} f_3$$

$$F_4 = \left(1-\frac{v^2}{c^2}\right)^{-1/2} i\dot{m}c.$$

Notes Prepared by
SAQIB HUSSAIN
M. Sc (Mathematics)

1996 (61)

Q Show that V_μ and F_μ are orthogonal.

Proof

We know that

$$V_\mu = \left(1-\frac{v^2}{c^2}\right)^{-1/2}(v, ic)$$

$$V_\mu \cdot V_\mu = \left(1-\frac{v^2}{c^2}\right)^{-1}(v^2-c^2)$$

$$\Rightarrow V_\mu^2 = \left(\frac{c^2}{c^2-v^2}\right)\cdot(v^2-c^2)$$

$$V_\mu^2 = -c^2$$

Diff w.r.t τ:

$$2V_\mu \cdot \frac{dV_\mu}{d\tau} = 0$$

$$V_\mu \frac{d}{d\tau}(mV_\mu) = 0$$

$$V_\mu \frac{dP_\mu}{d\tau} = 0$$

$$V_\mu \cdot F_\mu = 0$$

$\Rightarrow$ V_μ and F_μ are orthogonal

1997

Kinetic energy

Since we know that

$$V_\mu \cdot F_\mu = 0$$

$$\left(1-\frac{v^2}{c^2}\right)^{-1/2}(v, ic) \cdot \left(1-\frac{v^2}{c^2}\right)^{-1/2}(f, i\dot{m}c) = 0$$

$$\left(1-\frac{v^2}{c^2}\right)^{-1}\left[(v, ic) \cdot (f, i\dot{m}c)\right] = 0$$

$$v \cdot f - \dot{m}c^2 = 0 \quad \text{—— (1)}$$

But v.f is rate of doing work then

$$T = W = \int \dot{m}c^2 dt$$

$$= c^2 \int dm \quad \because \dot{m} = \frac{dm}{dt}$$

$$W = c^2 m + A \quad \text{—— (2)}$$

when v = 0 then T = 0

$$\Rightarrow 0 = c^2 m_0 + A$$

(2) ⇒

$$T = mc^2 - m_0 c^2 \quad \text{—— (*)}$$

$$= \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2$$

$$= m_0 c^2 \left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

$$= m_0 c^2 \left[1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots - 1\right]$$

$$= m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2}\right)$$

$$= \frac{1}{2} m_0 v^2$$

which is the classical expression for kinetic energy.

Remark

A particle of mass m is supposed to possess an energy mc^2

$$E = mc^2$$

from ⊛

$$mc^2 = T + m_0 c^2$$

$$E = T + m_0 c^2$$

If the particle is at rest then $E = m_0 c^2$ which is the internal energy of particle. The above equation implies that if a particle

is converted into energy radiation i.e electromagnetic radiation. The energy released is $E = mc^2$

Lorentz Transformation For four force components

$$v \cdot f - c^2 \dot{m} = 0$$

$$c^2 \dot{m} = v \cdot f$$

$$i c \dot{m} = \frac{i}{c} v \cdot f$$

As the relation for four force vector is

$$F_\mu = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} (f, i \dot{m} c)$$

$$= \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \left(f, \frac{i}{c} v \cdot f\right)$$

$$F_1 = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} f_x$$

$$F_2 = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} f_y$$

$$F_3 = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} f_z$$

$$F_4 = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \frac{i}{c} v \cdot f$$

The transformation equation for the Component of force are

$$\bar{F}_1 = F_1 \cos\alpha + F_4 \sin\alpha$$

$$\bar{F}_2 = F_2 \quad , \quad \bar{F}_3 = F_3 \qquad \text{③}$$

$$\bar{F}_4 = -F_1 \sin\alpha + F_4 \cos\alpha$$

The above transformation equation Can be abtained by differentiating Component of $\bar{P}_4$ w.r.t. τ.

Equation ③ are equivalent to

~~Doppler effect in Relativity~~

Motion of Particle with variable mass.

Let us consider m_o with proper mass of the particle

As $F_\mu = \frac{d}{d\tau}(m_o V_\mu)$

$$= m_o \frac{dV_\mu}{d\tau} + V_\mu \frac{dm_o}{d\tau} \quad \text{--- (1)}$$

we know that

$$V_\mu \cdot V_\mu = -c^2$$

$$V_\mu^2 = -c^2$$

$$\Rightarrow 2V_\mu \cdot \frac{dV_\mu}{d\tau} = 0 \quad \text{--- (2)}$$

Now

$$F_\mu \cdot V_\mu = m_o V_\mu \cdot \frac{dV_\mu}{d\tau} + V_\mu^2 \frac{dm_o}{d\tau}$$

$$F_\mu \cdot V_\mu = 0 + V_\mu^2 \frac{dm_o}{d\tau}$$

$$F_\mu \cdot V_\mu = -c^2 \frac{dm_o}{d\tau} \quad \text{--- (3)}$$

$\Rightarrow$ F_μ and V_μ are no more orthogonal

$$V_\mu \cdot F_\mu = \left(1-\frac{v^2}{c^2}\right)^{-1/2}(v, ic) \cdot \left(1-\frac{v^2}{c^2}\right)^{-1/2}(f, i\dot{m}c)$$

$$V_\mu \cdot F_\mu = \left(1-\frac{v^2}{c^2}\right)^{-1}\left(v \cdot f - \dot{m}c^2\right) \quad \text{--- (9)}$$

$$\Rightarrow -c^2 \frac{dm_0}{d\tau} = \left(1-\frac{v^2}{c^2}\right)^{-1}\left(v \cdot f - \dot{m}c^2\right)$$

$$\Rightarrow -c^2\left(1-\frac{v^2}{c^2}\right)\frac{dm_0}{d\tau} = v \cdot f - \frac{d}{dt}(mc^2)$$

$$-c^2\left(1-\frac{v^2}{c^2}\right)\frac{dm_0}{d\tau} = v \cdot f - \frac{dE}{dt}$$

$$\Rightarrow \frac{dE}{dt} = v \cdot f + c^2\left(1-\frac{v^2}{c^2}\right)\frac{dm_0}{d\tau}$$

$$\frac{dE}{dt} = v \cdot f + (c^2 - v^2)\frac{dm_0}{d\tau}$$

This is the rate of change of energy due to external source.

GENTLEMAN

Particles of Zero rest mass.

We know that the energy E and momentum P of a particle are related by the equation

$$E^2 = m_0^2 c^4 + p^2 c^2 \quad \text{---} \quad (1)$$

Also we have

$$E = mc^2 \text{ and } P = mV$$

Eliminating 'm' from these two equations we have

$$\frac{E}{P} = \frac{c^2}{V} \quad \text{---} \quad (2)$$

If a particle is moving with velocity 'c' then $V = c$.

$\therefore$ (2) $\Rightarrow$

$$\frac{E}{P} = c$$

$$\Rightarrow E = Pc$$

Then from equation (1)

$$E^2 = m_0^2 c^4 + E^2$$

$$\Rightarrow m_0^2 c^4 = 0$$

$$\Rightarrow m_0 = 0$$

this equation shows that the rest mass of a particle moving with the velocity of light must be zero.

Photons and Neutrinos are such particles. It must however be stressed that such particles actually can never come to rest. In fact if a particle is moving with velocity c in one frame it would be moving with the same velocity 'c' in all inertial frame.

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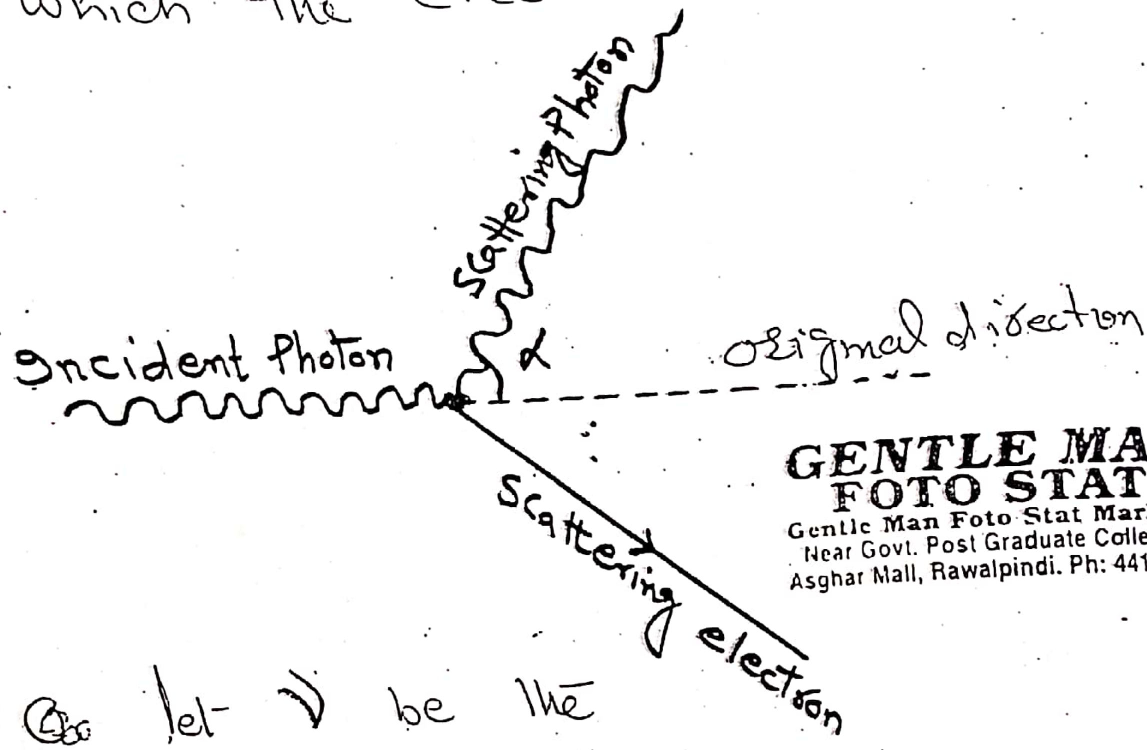
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Compton Effect

$E = h\nu$

When electromagnatic radiation of high frequency and Consequently of high energy is incident on electrons of a light element in which the electrans are losely bound to the nucleaus and Can be treated as free. The Scattered rays are found to have smaller frequency in addition to the radiation of original frequency. This is known as the Comptan effect. The change in the frequency of the incident radiation is independent of its initial frequency and dependents upon the angle of scattering. Consider the Scattering of a Photon of frequency

ν on an electron of mass m_0 in a frame of reference in which the electron is at rest.



Let ν' be the frequency of scattering photon let p be a momentum of the electron after the collision and the angle b/w the final and initial θ direction of the photon, this is shown in the figure.

Since the incident radiation is of high frequency the ~~photon~~ incoming

Photon by virtue of relation $E = h\nu$ is very energatic ∴ the electron which is lightly bound to its atom may be considered as free.

thus the Compton effect may be treated as a problem of collision b/w a photon and a stationary free electron.

the initial energy of photon is $h\nu_0$ and its initial momentum is $\frac{h\nu_0}{c}$ while its final energy and momentum are $h\nu$ and $\frac{h\nu}{c}$ respectively.

According to law of conservation of energy we have

$$h\nu_0 + m_0 c^2 = h\nu + E \quad \text{— (1)}$$

$$\frac{h\nu_0}{c} \underline{a} = \frac{h\nu}{c} \underline{b} + P \quad \text{— (2)}$$

where $\underline{a}$ and $\underline{b}$ are

unit vectors in the direction of the incident and scattered photon and E is the energy of the electron after callision.

From equation (1)

$$h\nu_0 - h\nu = E - m_0c^2$$

Squaring both sides

$$h^2\nu_0^2 + h^2\nu^2 - 2h^2\nu\nu_0 = E^2 + m_0^2c^4 - 2Em_0c^2 \quad (3)$$

from equation (2)

$$h\nu_0 a - h\nu b = cP$$

Squaring both sides

$$h^2\nu_0^2 + h^2\nu^2 - 2h^2\nu\nu_0\cos\alpha = c^2p^2 \quad (4)$$

Since

$$E^2 = m_0^2c^4 + p^2c^2$$

$$\Rightarrow c^2p^2 = E^2 - m_0^2c^4$$

Equation (4) ⇒

$$h^2\nu_0^2 + h^2\nu^2 - 2h^2\nu\nu_0\cos\alpha = E^2 - m_0^2c^4 \quad (5)$$

subtracting (3) and (5)

$$2h^2\nu\nu_0 - 2h^2\nu\nu_0\cos\alpha = -2m_0^2c^4 + 2Em_0c^2$$

$$2h^2\nu_0\nu(1-\cos\alpha) = 2m_0c^2(E - m_0c^2)$$

$$2h^2\nu_0\nu = 2m_0c^2(h\nu_0 - h\nu)$$ using (1)

$$\Rightarrow \frac{h}{m_0c^2}(1-\cos\alpha) = \frac{\nu_0-\nu}{\nu_0\nu}$$

$$\Rightarrow \frac{1}{\nu} - \frac{1}{\nu_0} = \frac{h}{m_0c^2}(1-\cos\alpha)$$

As $\lambda = \frac{c}{\nu}$ $(\because c = \nu\lambda)$

$$\Rightarrow \frac{c}{\nu} - \frac{c}{\nu_0} = \frac{h}{m_0c}(1-\cos\alpha)$$

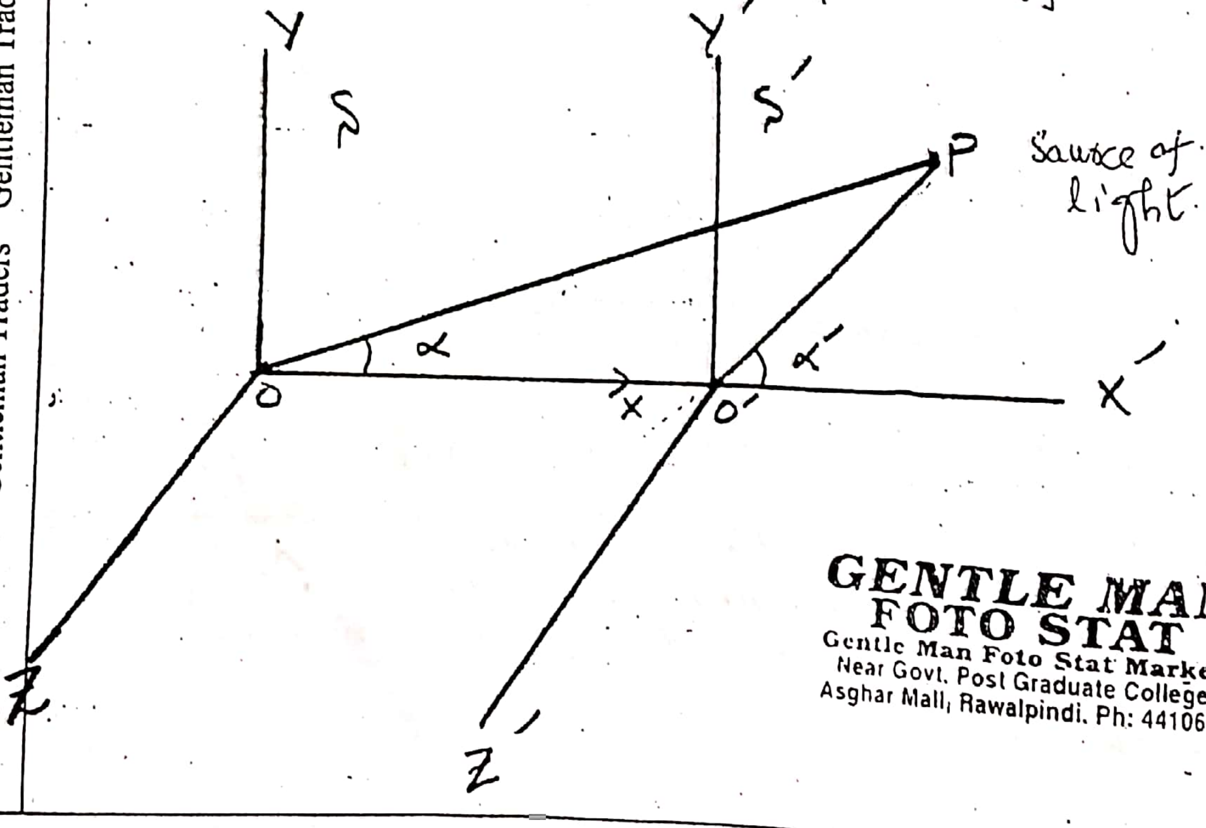
$$\lambda - \lambda_0 = \frac{h}{m_0c}(1-\cos\alpha)$$

$$\lambda - \lambda_0 = \frac{2h}{m_0c}\sin^2\frac{\alpha}{2}$$

Where λ_0 is initial wave length & λ is final wave length. And difference of λ and λ_0 is equal to change in wave length

Doppler effect in relativity

The change in the frequency of light radiation emitted by a source due to the relative motion of the source and the observer is called the Doppler effect. For light radiation we will use the transformation laws for energy and momentum to derive the relativistic formula for Doppler effect.



let S and S' be two frame of reference such that S' is moving relative to S with the velocity V along the positive direction of x-axis.

Suppose that a source P of monochromatic radiation is at rest in S' and is emitting radiation of frequency ν' as measured by an observer at the origin O' of S'. Then according to this observer the energy and magnitude of momentum of the photon would be given by $E' = h\nu'$ and $p = \frac{h\nu'}{c}$

If ν is the frequency of the same radiation as measured by an observer at the origin O of S then according to him the

energy and the magnitude of the momentum of photon would be $E = h\nu$ and

$$p = \frac{h\nu}{c} \quad \text{respectively.}$$

$$m' = \frac{m - vp_x/c^2}{\sqrt{1 - v^2/c^2}}$$

$$m'c^2 = \frac{mc^2 - vp_x}{\sqrt{1 - v^2/c^2}} \quad \text{—— (1)}$$

$$\Rightarrow E' = \gamma (E - vp_x) \quad \text{—— (2)}$$

where, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

- The inverse transformation is obtained by changing the sign of v and interchanging the primed and unprimed ~~Coordinates~~ quantities. Thus we have

$$E = \gamma (E' + vp'_x) \quad \text{—— (2)}$$

If α' is the angle which the line $\vec{op}$ make with the direction of x'-axis then

p'_x = the x-component of the momentum of the photon in the S' frame

$$p_x' = \frac{h\nu'}{c}\cos(\pi + \alpha') \quad \left(\because \text{light is propagated in the direction of } p_0'\right)$$

$$p_x' = -\frac{h\nu'}{c}\cos\alpha'$$

So equation (2) becomes

$$h\nu = \gamma\left(h\nu' + v\left(-\frac{h\nu'}{c}\cos\alpha'\right)\right)$$

$$\nu = \gamma\nu'\left(1 - \frac{v}{c}\cos\alpha'\right) \quad \text{--- (3)}$$

where ν' is the proper frequency of light. This formula gives the frequency ν' of a light source which is moving with the velocity v relative to the observer in S.

The change in frequency due to relative motion between the source and the observer is called relativistic Doppler shift.

Note that in classical physis there are two formulas for Doppler effect one for the moving source and and other for moving

observer. In special theory of relativity we have only one formula for the Doppler effect. Because only the relative motion b/w the source and the observer is sufficient: the inverse transformation is obtained by changing the sign of V and interchanging prime and unprime quantities

$$\nu' = \gamma \nu \left(1 + \frac{V}{c} \cos\alpha\right) \quad \text{—(8)}$$

where α is the angle which the line $\overrightarrow{OP}$ makes with the x-axis.

Special Cases :-

The Longitudinal Doppler effect :-

when the source P is receding directly away from the observer. O

$\therefore \quad \alpha' = 0$

As $\quad \nu = \gamma \nu' \left(1 - \frac{V}{c} \cos\alpha'\right)$

$$\nu = \gamma \nu' \left(1 - \frac{V}{c}(1)\right)$$

$$\Rightarrow \quad \nu = \frac{\nu' (1 - V/c)}{\sqrt{1 - V^2/c^2}}$$

$$\nu = \nu' \sqrt{\frac{1-v/c}{1+v/c}} \quad \text{--- (5)}$$

Since $\sqrt{\frac{1-v/c}{1+v/c}} < 1$

$\therefore \quad \nu < \nu'$

$\Rightarrow$ the frequency ν is less than ν' i.e when the source moving away from the observer.

When the source is moving directly towards the observer then

$\alpha' = \pi$

then (3) $\Rightarrow$

$$\nu = \nu' \sqrt{\frac{1+v/c}{1-v/c}} \quad \text{--- (6)}$$

$\Rightarrow \quad \nu > \nu'$

<u>Case 2</u> <u>The Transverse Doppler effect for</u> of $\alpha' = \frac{\pi}{2}$

Then Eq (3) $\Rightarrow$

$$\nu = \gamma . \nu'$$

This equation shows that even if the observervation is made at right angle to the direction of motion of a source a change in the frequency of light radiation occures. No such effect is predicted by classical theory for which $\nu = \nu'$ when $\alpha' = \frac{\pi}{2}$

Realtivity Problems

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Q1 Let S and S′ are two Galillian frames. S′ is moving along common x-axis with velocity v. A particle is observed by frame S′ describes a circle

$$x'^2 + y'^2 = a^2 , \quad z' = 0$$

with a constant speed. Show that as observed from frame S the particle appears to move in an ellipse where centre move with speed v.

Sol By Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y , \quad z' = z$$

As $x'^2 + y'^2 = a^2$

$$\therefore \left(\frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \right)^2 + y^2 = a^2$$

$$\frac{(x - vt)^2}{\left(a\sqrt{1 - \frac{v^2}{c^2}} \right)^2} + \frac{y^2}{a^2} = 1$$

This is the equation of an ellipse in xy-plane whose centre moves along x-axis with velocity $\underline{v}$

Q2 obtain the transformation for the component of $\underline{v}$ by differentiating Lorentz transformation.

Soln

As we know that Lorentz transformation are

$$\bar{x} = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\bar{y} = y \,, \quad \bar{z} = z$$

$$\bar{t} = \frac{t - \frac{u}{c^2}x}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Diff w.r.t $\bar{t}$

$$\frac{d\bar{x}}{d\bar{t}} = \frac{d}{dt}\left(\frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}}\right)\frac{dt}{d\bar{t}} \quad (1)$$

As $\bar{t} = \dfrac{t - \frac{u}{c^2}x}{\sqrt{1-\frac{u^2}{c^2}}}$

$$\frac{d\bar{t}}{dt} = \frac{1 - \frac{u}{c^2}v_x}{\sqrt{1-\frac{u^2}{c^2}}}$$

So ① ⇒

$$\frac{d\bar{x}}{d\bar{t}} = \frac{\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{u}{c^2}v_x}\left(\frac{v_x - u}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$\bar{V}_x = \frac{V_x - u}{1-\frac{u}{c^2}v_x}$$

$$\bar{V}_y = \frac{d\bar{y}}{d\bar{t}} = \frac{dy}{dt}\cdot\frac{dt}{d\bar{t}}$$

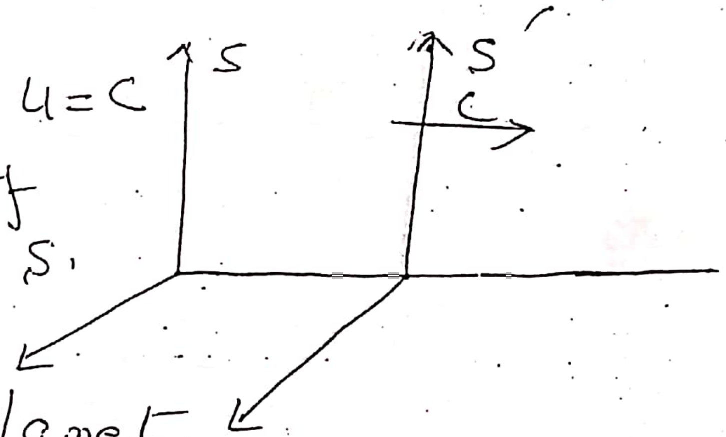
$$= \frac{V_y\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{uv_x}{c^2}}$$

$$\bar{V}_z = \frac{V_z\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{uv_x}{c^2}}$$

Q3 obtain the transformation for the Components of $\underline{a}$ by differentiating Lorentz transformation.

Q4 If S' has velocity c relative to S. Show that all planets moving relative to S with velocity less than c have a velocity c relative to S'

Soln



Here $u = c$

i.e velocity of S' relative to S.

let $\underline{V}$ be the velocity of Planet relative to S.

i.e $V < c$

let V_x', V_y', V_z' be the Velocities of The planet relative to S' we know that

$$V_x' = \frac{V_x - u}{1 - \frac{uV_x}{c^2}}$$

Here $u = c$

$$\therefore V_x' = \frac{V_x - c}{1 - \frac{V_x}{c}} = -c$$

$$\Rightarrow |V_x'| = |-c| = c$$

$$V_y' = \frac{V_y\sqrt{1 - \frac{u^2}{c^2}}}{1 - \frac{uV_x}{c^2}}$$

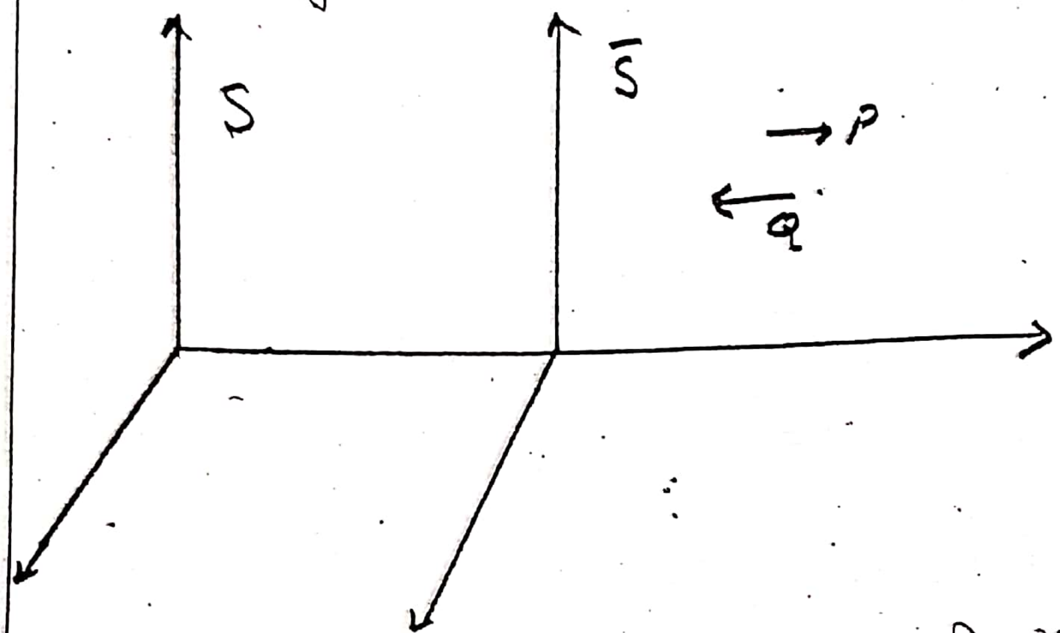
$$V_y' = 0$$

Similarly $V_z' = 0$

Q5. Two points are moving in opposite direction with velocity c relative to some inertial frame. Show that there relative speed is c.

Sol: Let the velocity of the point 'p' is c along x-axis and the point q be $-c$

along the same axis



Suppose that P is fixed in frame $\bar{S}$. velocity of Q relative to P is same as its velocity relative to S,

Since

$$\bar{V}_x = \frac{V_x - u}{1 - \frac{uV_x}{c^2}}$$

As $u = c$

$$V_x = -c$$

$$\therefore \bar{V}_x = \frac{-c - c}{1 - \frac{c(-c)}{c^2}}$$

$$V_x = \frac{-2C}{2} = -C$$

$$|\bar{V}_x| = |-c| = c$$

Also $\bar{V}_y = \bar{V}_z = 0$

Hence there relative velocity is C.

Q6 Show that energy transformation according to Lorentz transformation is given by

$$\bar{E} = \frac{E - P_x u}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Sol: As we know that

$$\bar{m} = \frac{m - muV_x/c^2}{\sqrt{1 - u^2/c^2}}$$

xlying both sides by c^2

$$\bar{m}c^2 = \frac{mc^2 - muV_x}{\sqrt{1 - u^2/c^2}}$$

$$\bar{E} = \frac{E - u P_x}{\sqrt{1 - u^2/c^2}} \quad \because P_x = mV_x$$

Q7 Show that four velocity vector $\bar{V}_\mu$ is of constant magnitude

Sol:n As we know that

$$\bar{V}_\mu = \frac{(v, ic)}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\bar{V}_\mu \cdot \bar{V}_\mu = \frac{(v, ic)}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \left(\frac{(v, ic)}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$= \frac{v^2 - c^2}{1-\frac{v^2}{c^2}}$$

$$\Rightarrow V_\mu^2 = -c^2$$

$$\Rightarrow |V_\mu| = ic$$

Four velocity vector is of constant magnitude.

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Q8) A radio station is fixed in a frame S and sends out waves. Show that for another observer in another frame S' these waves at any instant forms a concentric spheres family. Is it possible that two of these spheres might intersect.

Sol: let the radio station relative to frame S fixed at position (a, 0, 0). Now wave front relative to S at any instant t is

$$(x-a)^2 + y^2 + z^2 = c^2t^2 \quad \text{— (i)}$$

Transformation laws are

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$y = y', \quad z = z'$$

$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - v^2/c^2}}$$

$\therefore$ (1) $\Rightarrow$

$$\left(\frac{x'+vt'}{\sqrt{1-v^2/c^2}}-a\right)^2+y'^2+z'^2=c^2\left[\frac{t'+\frac{v}{c^2}x'}{\sqrt{1-v^2/c^2}}\right]^2$$

thus relative to an observer in S' frame - these waves form a family of concentric spheres.

It is not possible for two wave fronts to intersect because intersection means that one and the same wave-front is visible to the observer at two different instants which is physically impossible.

Q8 A man walks to work at a speed of 6 km/h. How accurately would it be necessary to measure his walking stick to be able to detect the Lorentz-Fitzgerald contraction of the

Stick if it is 1m long. 98 such an accuracy physically attainable.

Soln Velocity $= v = 6\,km/h$

$= \frac{6\times 1000}{3600} = 1.66\,m/sec$

velocity of light $= c = 3\times 10^8\,m/sec$

Length of stick $= l = 1\,m$

$\Delta l = l - l' = ?$

As we know that

$l' = l\sqrt{1-\frac{v^2}{c^2}} = l\left(1-\frac{v^2}{c^2}\right)^{1/2}$

$= l\left(1-\frac{1}{2}\frac{v^2}{c^2}\right)$ (Neglecting Small terms)

Now $\Delta l = l - l'$

$= l - l\left(1-\frac{v^2}{2c^2}\right)$

$= l\left[1-1+\frac{v^2}{2c^2}\right] = l\left(\frac{v^2}{2c^2}\right)$

$\Delta l = \frac{(1.66)^2}{2\times(3\times10^8)^2} = 1.53\times10^{-17}\,m$

Since Δl is very Small therefore it is not physically attainable

Q9 An electronic device capable of measuring changes of 10^{-2} cm is being used at a fencing match and it sees a 1m foil decreased in size. Considering the matter quantitatively could the decrease be due to Lorentz-Fitzgerald Contraction

Sol:

Length of foil = $l = 1m$

Change in length of foil $= \Delta l = 10^{-2} cm$

$= 10^{-4} m$

Velocity of light $= c = 3 \times 10^{8} m/sec$

As $\Delta l = l - l'$

$\Rightarrow l' = -\Delta l + l$

$= -10^{-4} + 1$

$l' = 0.9999 m$

As we know that

$l' = l\sqrt{1 - \frac{v^2}{c^2}}$

$\frac{l'}{l} = \sqrt{1 - \frac{v^2}{c^2}}$

$\left(\frac{l'}{l}\right)^2 = 1 - \frac{v^2}{c^2}$

$$\frac{V^2}{C^2} = 1 - \left(\frac{l'}{l}\right)^2$$

$$\frac{V}{C} = \sqrt{1 - \frac{l'^2}{l}}$$

$$\frac{V}{c} = \sqrt{1 - \frac{(0.9999)^2}{1}}$$

$$\frac{V}{c} = \sqrt{1.0}$$

$$\frac{V}{c} = 0.0141$$

$$V = 0.0141C$$

$$= 0.0141 \times 3 \times 10^8$$

$$= 4230 \text{ km/sec}$$

Since V is very large - Therefore this Contraction is not due to Lorentz - Fitzgerald Contraction.

Q10 what are the percentage changes in length due to the Lorentz - Fitzgerld Contraction for the following.

(a) A train moving with speed 60km/h

(b). A Car moving with Speed 300km/h

(c) A ~~car~~ jetplane moving with speed 600 km/h

(d) A satellite moving with speed 28,600 km/h

(e) A deep-space reconassiana vehicle moving with speed 93,900 km/h

(f) the earth moving round the sun with speed 30,000 km/hr

Sol/n ~~the earth moving round the sun~~

As we know that

$$l' = l\sqrt{1-\frac{v^2}{c^2}}$$

$$\frac{l'}{l} = \sqrt{1-\frac{v^2}{c^2}} = \left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

$$\frac{l'}{l} = 1-\frac{1}{2}\frac{v^2}{c^2}$$

$$\frac{v^2}{2c^2} = 1-\frac{l'}{l}$$

$$\frac{v^2}{2c^2} = \frac{l-l'}{l}$$

$$\frac{v^2}{2c^2} = \frac{\Delta l}{l}$$

therefore percentage change in

length due to Lorentz-Fitzgerald Contraction is

$$\frac{\Delta l}{l} \times 100 = \frac{v^2}{2c^2} \times 100 = 50\frac{v^2}{c^2}$$

(a) <u>For a train:-</u>

$$v = 60 \text{ km/h} = \frac{60 \times 1000}{3600} = \frac{50}{3} \text{ m/sec}$$

$$c = 3 \times 10^8 \text{ m/sec}$$

$$\% \text{ change in length} = \frac{50\left(\frac{50}{3}\right)^2}{(3 \times 10^8)^2}$$

$$= \frac{(50)^3}{3^4 \times 10^{16}}$$

$$= 1.54 \times 10^{-13} \%$$

(b) <u>For a Racing Car</u>

$$v = 300 \text{ km/h} = \frac{300 \times 1000}{3600} = \frac{250}{3} \text{ m/sec}$$

$$= 83.33 \text{ m/sec}$$

$$c = 3 \times 10^8 \text{ m/sec}$$

$$\% \text{ change in length} = \frac{50(83.33)^2}{(3 \times 10^8)^2}$$

$$= 3.86 \times 10^{-12} \%$$

Q11 The History of moving Particles is represented in space diagram by Hyperbola

$$\frac{x^2}{c^2} - \frac{t^2}{1} = \frac{1}{k^2}$$

Show That $\frac{d^2x}{ds^2} = k^2 x$ and $\frac{d^2t}{ds^2} = k^2 t$

where s is the proper time.

Sol: The Parametric equations of the Hyperbola are

$$x = \frac{c}{k}\cosh\theta \quad , \quad t = \frac{1}{k}\sinh\theta$$

Now $\frac{dx}{ds} = \frac{dx}{d\theta} \cdot \frac{d\theta}{ds}$ ——— (1)

Since we know that

$$d\tau^2 = (dt)^2 - \frac{1}{c^2}\sum_{i=1}^{3}(dx_i)^2$$

As $d\tau \longrightarrow ds$

$$\therefore (ds)^2 = (dt)^2 - \frac{1}{c^2}\sum_{i=1}^{3}(dx_i)^2$$

$$\Rightarrow \left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dt}{d\theta}\right)^2 - \frac{1}{c^2}\sum_{i=1}^{3}\left(\frac{dx_i}{d\theta}\right)^2$$

Now as $t = \frac{1}{k}\sinh\theta$

$$\frac{dt}{d\theta} = \frac{1}{k}\cosh\theta$$

$$x = \frac{c}{k}\cosh\theta$$

$$\frac{dx}{d\theta} = \frac{c}{k}\sinh\theta$$

$$\therefore \left(\frac{ds}{d\theta}\right)^2 = \frac{1}{k^2}\left(\cosh^2\theta - \sinh^2\theta\right)$$

$$= \frac{1}{k^2}$$

$$\Rightarrow \frac{ds}{d\theta} = \frac{1}{k}$$

$$\frac{d\theta}{ds} = k$$

$\therefore$ (1) $\Rightarrow$

$$\frac{dx}{ds} = k \cdot \frac{c}{k}\sinh\theta$$

$$\frac{dx}{ds} = c\sinh\theta$$

$$\frac{d^2x}{ds^2} = \frac{d}{d\theta}\left(\frac{dx}{ds}\right)\frac{d\theta}{ds}$$

$$= k \cdot c\cosh\theta$$

$$= K \cdot xK = K^2 x$$

$$\frac{dt}{ds} = \frac{dt}{d\theta} \cdot \frac{d\theta}{ds} \quad \text{—— (2)}$$

$$= K \cdot \frac{1}{K} \cosh\theta$$

$$\frac{dt}{ds} = \cosh\theta$$

$$\frac{d^2t}{ds^2} = \frac{d}{d\theta}\left(\frac{dt}{ds}\right)\frac{d\theta}{ds}$$

$$= K \cdot \sinh\theta$$

$$= K \cdot Kt$$

$$\frac{d^2t}{ds^2} = K^2 t$$

As required.

Q 12. An Aeroplane sets up to fly at 500 miles per hour. Show that it would have to fly for more than one thousands year in order to make a difference of 1/100 seconds between the time recorded by a clock in the aeroplane and a

clock on the ground.

Sol'n

From Lorentz Transformation we have

$$t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1-v^2/c^2}}$$

$t_1 = t_1' = 0$

$$\Rightarrow t_2 = \frac{t_2'}{\sqrt{1-\frac{v^2}{c^2}}}$$

where 'v' is the speed of aeroplane relative to the ground and t_2' is the time recorded on the aeroplane.

$$\text{Now } t_2 - t_2' = \frac{t_2'}{\sqrt{1-\frac{v^2}{c^2}}} - t_2'$$

$$= t_2'\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

$$= t_2'\left[1 + \frac{v^2}{2c^2} - 1\right]$$ (using binomial series and neglecting higher powers of $\frac{v^2}{c^2}$)

$$t_2 - t_2' = \frac{t_2' v^2}{2c^2}$$

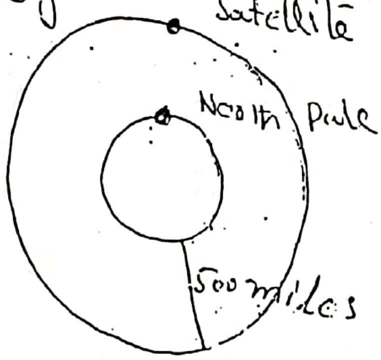
$$\frac{1}{100} = t_2' \cdot \frac{500 \times 1760 \times 3}{3600 \times 2 \times 9 \times 10^8}$$

$$t_2' = \quad > 1000$$

As required

Q13 Two identical clocks C_1 and C_2 are Constructed. C_1 is placed at north pole and C_2 is placed at an artifical satelite which is made to describe a circular orbit at 500 miles above the earth and passing over the pole. If E_1 and E_2 are two events Corresponding to the successive Passages of satellite over the north pole. Find the radio of time interval b/w E_1 and E_2.

Soln As measured by C_1 and C_2 the velocity of the artifical satellite in its orbit is given by



$$V = \sqrt{\frac{gR^2}{r}}$$

$g = 32 \text{ ft}/\text{sec}^2$

R = radius of earth = 4000 miles = 4000 × 1760 × 3

R =

$$V = \sqrt{\frac{32(\quad)^2}{500 \times 176 \times 3}}$$

$= 7.352 \times 10^4 \text{ ft}/\text{sec}$

Q14 Two electrons move towards each other with the speed of each being 0.9c in a Galillian frame of reference. what are there speeds relative to each other.

Sol'n let e_1 and e_2 be two electrons and there relative velocity to a frame S_1 be 0.9c and −0.9c along x-axis

$$\therefore \quad V_x = 0.9c$$

$$\& \quad u = -0.9c$$

$$\bar{V}_x = \frac{V_x - u}{1 - \frac{uV_x}{c^2}}$$

$$= \frac{0.9c + .9c}{1 + \frac{(0.9)c^2}{c^2}} = 0.994c$$

Q15 Two particles with proper masses m_1 and m_2 move along x-axis in a frame S with velocities u_1 and u_2 respectively. They collide and collase to form a single particle. Prove that the proper mass m_3 and velocity u_3 is given by

$$m_3^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2\left(1 - \frac{u_1u_2}{c^2}\right)$$

and $$u_3 = \frac{m_1\gamma_1u_1 + m_2\gamma_2u_2}{m_1\gamma_1 + m_2\gamma_2}$$

where $\gamma_1 = \frac{1}{\sqrt{1-u_1^2/c^2}}$, $\gamma_2 = \frac{1}{\sqrt{1-u_2^2/c^2}}$

Sol'n

From the law of Conservation of energy we have

$$\frac{m_1c^2}{\sqrt{1-\frac{u_1^2}{c^2}}} + \frac{m_2c^2}{\sqrt{1-\frac{u_2^2}{c^2}}} = \frac{m_3c^2}{\sqrt{1-\frac{u_3^2}{c^2}}}$$

$$\Rightarrow m_1\gamma_1 + m_2\gamma_2 = m_3\gamma_3 \quad \text{---} (1)$$

By the law of Conservation of momentum

$$\frac{m_1 u_1}{\sqrt{1-\frac{u_1^2}{c^2}}} + \frac{m_2 u_2}{\sqrt{1-\frac{u_2^2}{c^2}}} = \frac{m_3 u_3}{\sqrt{1-\frac{u_3^2}{c^2}}}$$

$$\Rightarrow m_1 \gamma_1 u_1 + m_2 \gamma_2 u_2 = m_3 \gamma_3 u_3 \quad \text{--- (2)}$$

From Equation (1)

$$m_3 = \frac{m_1 \gamma_1 + m_2 \gamma_2}{\gamma_3}$$

$$m_3 = \left(1 - \frac{u_3^2}{c^2}\right)^{1/2} (m_1 \gamma_1 + m_2 \gamma_2)$$

$$m_3^2 = \left(1 - \frac{u_3^2}{c^2}\right) (m_1 \gamma_1 + m_2 \gamma_2)^2 \quad \text{--- } (*)$$

Now from Equation (2)

$$u_3 = \frac{m_1 \gamma_1 u_1 + m_2 \gamma_2 u_2}{m_3 \gamma_3}$$

$$= \frac{m_1 \gamma_1 u_1 + m_2 \gamma_2 u_2}{m_1 \gamma_1 + m_2 \gamma_2} \quad \text{(Using (1))}$$

∴ Eq'n $(*) \Rightarrow$

$$m_3^2 = \left[1 - \frac{(m_1\gamma_1u_1 + m_2\gamma_2u_2)^2}{c^2(m_1\gamma_1 + m_2\gamma_2)^2}\right](m_1\gamma_1 + m_2\gamma_2)^2$$

$$m_3^2 = \left[\frac{c^2(m_1\gamma_1 + m_2\gamma_2)^2 - (m_1\gamma_1u_1 + m_2\gamma_2u_2)^2}{c^2(m_1\gamma_1 + m_2\gamma_2)^2}\right](m_1\gamma_1 + m_2\gamma_2)^2$$

$$m_3^2 = m_1^2\gamma_1^2 + m_2^2\gamma_2^2 + 2m_1m_2\gamma_1\gamma_2 - \frac{1}{c^2}(m_1^2\gamma_1^2u_1^2 + m_2^2\gamma_2^2u_2^2 + 2m_1m_2\gamma_1\gamma_2u_1u_2)$$

$$m_3^2 = m_1^2\gamma_1^2\left(1 - \frac{u_1^2}{c^2}\right) + m_2^2\gamma_2^2\left(1 - \frac{u_2^2}{c^2}\right) + 2m_1m_2\gamma_1\gamma_2\left(1 - \frac{u_1u_2}{c^2}\right)$$

$$= m_1^2\gamma_1^2 \cdot \frac{1}{\gamma_1^2} + m_2^2\gamma_2^2 \cdot \frac{1}{\gamma_2^2} + 2m_1m_2\gamma_1\gamma_2\left(1 - \frac{u_1u_2}{c^2}\right)$$

$$m_3^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2\left(1 - \frac{u_1u_2}{c^2}\right)$$

As required

Q16 1999 A particle of mass m_1 and speed v callide with a particle of rest mass m_2 which is stationary. After the collision two particle coalace. Assuming that

there is no radiation of energy. Show that the rest mass 'M' of the combined particle is given by

$$M^2 = m_1^2 + m_2^2 + \frac{2 m_1 m_2}{\sqrt{1-\frac{v^2}{c^2}}}$$

Sol'n let u be the velocity of combined particle of mass M. According to law of conservation of energy.

$$\frac{Mc^2}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m_1 c^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_2 c^2 \longrightarrow (i)$$

$$\frac{M}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m_1}{\sqrt{1-\frac{v^2}{c^2}}} + m_2$$

According two law of conservation of momentum

$$\frac{Mu}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m_1 v}{\sqrt{1-\frac{v^2}{c^2}}} \longrightarrow (2)$$

Dividing (2) by (1)

$$U = \frac{m_1 v}{m_1 + m_2\sqrt{1-\frac{v^2}{c^2}}} \quad \text{——— (3)}$$

Now squaring equation (2)

$$\frac{M^2 u^2}{1-\frac{u^2}{c^2}} = \frac{m_1^2 v^2}{1-\frac{v^2}{c^2}}$$

$$\Rightarrow M^2 = \frac{m_1^2 v^2 (c^2 - u^2)}{(c^2 - v^2) u^2}$$

$$M^2 = \frac{m_1^2 v^2}{c^2 - v^2} \cdot \frac{c^2 - \frac{m_1^2 v^2}{(m_1 + m_2\sqrt{1-v^2/c^2})^2}}{\frac{m_1^2 v^2}{(m_1 + m_2\sqrt{1-\frac{v^2}{c^2}})^2}}$$

$$= \frac{m_1^2 v^2}{c^2 - v^2} \cdot \frac{c^2(m_1 + m_2\sqrt{1-\frac{v^2}{c^2}})^2 - m_1^2 v^2}{m_1^2 v^2}$$

$$= \frac{1}{c^2 - v^2} \cdot \left[m_1^2 c^2 + m_2^2 c^2 \left(1 - \frac{v^2}{c^2}\right) + 2 m_1 m_2 \sqrt{1-\frac{v^2}{c^2}} - m_1^2 v^2 \right]$$

$$= \frac{1}{c^2 - v^2} \left[m_1^2 (c^2 - v^2) + m_2^2 (c^2 - v^2) + 2 m_1 m_2 \frac{\ldots}{\sqrt{1-v^2/c^2}} \right]$$

$$= m_1^2 + m_2^2 + 2 m_1 m_2 \cdot \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Q17 If Pakistan is moving past India at a speed of 2.5×10^{10} cm/sec
(a) India will think our rockets are shorter than they actually are. If they see the rocket as 3 m long, how long are they for us? (b) They also see us as doing things slow. If it takes 1 min to fire the rockets. How long does it seem to them? (c) If the rockets are travelling with speed of 8×10^5 cm/sec in the direction of motion. How fast will the Indians see them moving.

Soln

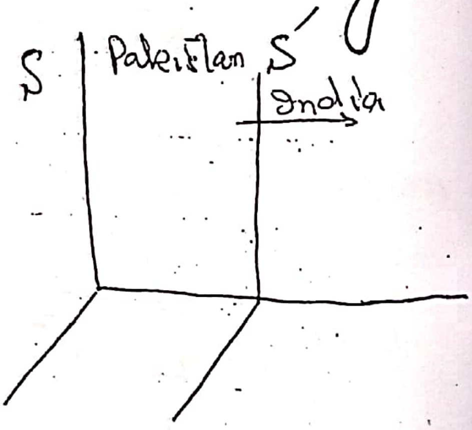
Given Data

$V = 2.5 \times 10^{10}$ cm/sec

$V = 2.5 \times 10^{8}$ m/sec

$C = 3 \times 10^{8}$ m/sec

(a) $l' = 3$ m



$l = ?$

By length Contraction formula

$$l' = l\sqrt{1-\frac{v^2}{c^2}}$$

$$\Rightarrow \quad l = \frac{l'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{3}{\sqrt{1-\left(\frac{2.5\times10^8}{3\times10^8}\right)^2}}$$

$$= \frac{3}{0.5528}$$

$$l = 5.4272\,m$$

(b) Given that

$t = 1\ min = 60\ Sec$

$t' = ?$

$$t' = \frac{t}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{60}{0.5528} = 108.54\ sec$$

$$\Rightarrow \quad t' = 108.54\ Sec$$

$$= 1\ min\ 49\ Sec$$

(C) Given that

$u = 8 \times 10^5 \, cm/sec$

$u = 8 \times 10^3 \, m/sec$

$u' = ?$

As we know that

$$u' = \frac{u + V}{1 + \frac{uV}{c^2}}$$

$$= \frac{8 \times 10^3 + 2.5 \times 10^8}{1 + \frac{(8 \times 10^3)(2.5 \times 10^8)}{(3 \times 10^8)^2}}$$

$$u' = 2.5 \times 10^8 \, m/sec$$

imp

Q18 An observer A, sees a body as having twice the length that another observer B sees. which of them has the greater speed relative to the body if it lies along the direction of their relative motion? If a third observer

sees the length as 3 times that seen by observer B and the body is in the rest frame of one of the three observers, which is the rest frame of the body.

Soln let l be the original length of the body and l_A is the length of the body seen by A and l_B is the length of body as seen by B. then

$$l_A = 2 l_B \Rightarrow V_A < V_B$$

let l_c be the length of the body as seen by C then

$$l_c = 3 l_B \Rightarrow V_c < V_B$$

For observer A :-

By length contraction formula

$$l' = l\sqrt{1 - v^2/c^2}$$

$$2l = l_0\sqrt{1 - V_A^2/c^2}$$

Squaring $4l^2 = l_0^2\left(1 - V_A^2/c^2\right)$.

$$V_A = C\sqrt{1-\frac{4l^2}{l_0^2}}$$

For observer B

$$l = l_0\sqrt{1-V_B^2/c^2}$$

Squaring $l^2 = l_0^2\left(1-V_B^2/c^2\right)$

$$V_B = C\sqrt{1-\frac{l^2}{l_0^2}}$$

For observer C

$$3l = l_0\sqrt{1-V_c^2/c^2}$$

$$\Rightarrow \quad 9l^2 = l_0^2\left(1-V_c^2/c^2\right)$$

$$\Rightarrow \quad V_c = C\sqrt{1-\frac{9l^2}{l_0^2}}$$

for checking the rest frame of the body we put $V_i = 0$ for $i = A, B, C$

we take here the value of l^2/l_0^2 and put this value in the other two velocities. for with which V_i the other two are reals.

that frame is the rest frame.

e.g. $V_A = 0$

$$\Rightarrow 1 - \frac{4l^2}{l_o^2} = 0$$

$$\frac{l^2}{l_o^2} = 1/4$$

with this V_B is real but V_C is imaginary. So it is not acceptable.

Now $V_B = 0$

$$\Rightarrow 1 - \frac{l^2}{l_o^2} = 0$$

$$\frac{l^2}{l_o^2} = 1$$

with this V_A and V_C are imaginary. So it is also not acceptable.

Now $V_C = 0$

$$\Rightarrow 1 - \frac{9l^2}{l_o^2} = 0$$

$$\frac{l^2}{l_o^2} = 1/9$$

with this V_A and V_B are real. therefore 'C' is the rest frame.

1999

Q19 A scientific setellite finds some cosmic radiation to be entirely composed of neutrans (which has a half life of about 1000 sec) coming at a speed of $(1-10^{-7})c$. Could the radiations have been produced outside the solar system (about 10^{10} km across) or must it originate from within the sollar system.

Soln let Δt be half life of neutrans in their rest frame and $\Delta t'$ be half life of neutrons as observed by setellite

$\therefore \Delta t = 1000$ sec

$\Delta t' = ?$ $V = (1-10^{-7})c$

$$\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \frac{1000}{\sqrt{1-(0.9999)^2}}$$

$\Delta t' = 2.36 \times 10^{6}$ sec

Now

$\Delta x'$ = Distance Covered by Cosmic radiations during the half life as observed by the satellite.

$\Delta x' = V \Delta t$

$= 0.9999\, c \times 2.236 \times 10^{6}$

$= 6.69 \times 10^{10}$ km $> 10^{10}$ km

This shows that the radiations have been produced outside the solar system.

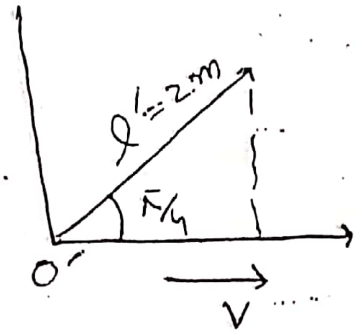
Q20 An observer sees a clock as showing 1 hour to be half an hour. If he sees an object lying at an angle of $\pi/4$ as having a length of 2 m what is the rest length of the object.

Sol'n

$l' = 2$ m

$l = ?$

$l_x' = ?\,,\ l_y' = ?$



$l_x' = l' \cos\frac{\pi}{4}$

$= 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}\,,\quad l_y' = 2\sin\frac{\pi}{4} = \sqrt{2}$

Since the motion is along x-axis

$\therefore\ l_y' = l_y \quad \& \quad l_x' = \frac{l_x}{\gamma}$

$\Rightarrow\ l_y = \sqrt{2} \qquad l_x = \gamma\, l_x'$

$l_x = \sqrt{2}\,\gamma .$

$\therefore\ l = \sqrt{l_x^2 + l_y^2}$

$= \sqrt{\quad}$

Since observes sees a clock as showing 1 hour to be half an hour.

$\therefore\ t = \frac{t'}{2} \Rightarrow \frac{t'}{t} = 2$

By time dilation

$\gamma = \frac{t'}{t} = 2$

$\left(\because\ t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}}\right.$

$t' = \gamma t$

$\Rightarrow \frac{t'}{t} = \gamma .$

$\therefore$ $l_x = 2\sqrt{2}$, $l_y = \sqrt{2}$

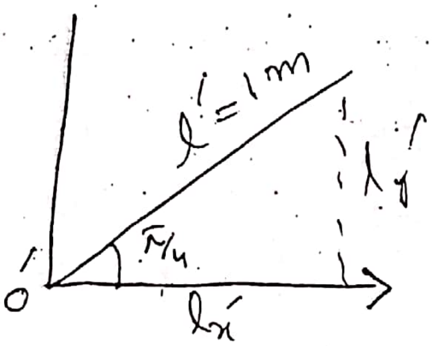
Thus $l = \sqrt{(2\sqrt{2})^2 + (\sqrt{2})^2}$

$= \sqrt{8+2} = \sqrt{10}$ m

Q21 An observer O′ sees a rod having a length of one meter and making an angle of $\pi/4$ with the direction of motion. If in its rest frame the rod makes an angle $\pi/6$ with the direction of motion. What is the rest length of the rod.

Sol:

w.r.t O′ frame



$l_y' = l' \sin \pi/4$

$= (1)\frac{1}{\sqrt{2}}$

$l_y' = \frac{1}{\sqrt{2}}$

w.r.t O

$l_y = l \sin \pi/6 = l(\frac{1}{2})$

$l_y = l/2$

Since the motion is along x-axis, so

$$l_y' = l_y$$

$$\frac{1}{\sqrt{2}} = \frac{l}{2}$$

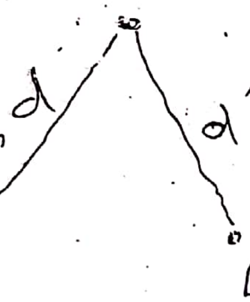
$$\Rightarrow \quad l = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$l = \sqrt{2}\ m$$

Q22

Two observers A and B see a rocket taking off at observed distances d and d′ and time t and t′ respectively, having their clocks at zero when they are coincident of their direction of relative motion is in line with the rocket. Find the speed of B relative to A.

Sol'n By Lorentz transformation



$$t' = \gamma\left(t - \frac{v}{c^2}d\right)$$

$$d' = \gamma(d - vt) \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \quad \frac{t'}{d'} = \frac{\gamma\left(t - \frac{v}{c^2}d\right)}{\gamma(d - vt)}$$

$$\frac{t'}{d'} = \frac{t - \frac{v}{c^2}d}{d - vt}$$

$$d'\left(t - \frac{v}{c^2}d\right) = t'(d - vt)$$

$$d't - \frac{v}{c^2}dd' = t'd - t'vt$$

$$\frac{dd'}{c^2}v - tt'v = d't - t'd$$

$$\left(\frac{dd'}{c^2} - tt'\right)v = d't - t'd$$

$$\left(\frac{dd' - c^2tt'}{c^2}\right)v = d't - t'd$$

$$v = \frac{c^2(d't - t'd)}{dd' - c^2tt'}$$

is the required speed

Q23 An observer O sees three bodies A, B, C have velocities $(u,0,0)$, $(0,-v,0)$, $(0,0,w)$ respectively. What is the velocity

of each body relative to other? In particular is the velocity of A as seen by B exactly opposite in direction and equal in magnitude to the velocity of B as seen by A. If not, way is there any difference. would the velocities be equal and opposite if the velocity of B is $(0, V, 0)$ instead of $(0, -V, 0)$?

Soln

$$\underline{V}_A = (u, 0, 0), \quad \underline{V}_B = (0, -V, 0)$$

$$\underline{V}_c = (0, 0, \omega)$$

let V_{AB}, V_{AC} be the velocities of A relative to B and C and $\underline{V}_{BA}$, V_{BC} be the velocities of B relative to A and C. and $\underline{V}_{CA}$, V_{CB} are velocities of C relative to A and B.

Since we know that

$$u' = \frac{\underline{u} + \underline{V}\left[\frac{\underline{u} \cdot \underline{V}}{V \cdot V}(\gamma - 1) - \gamma\right]}{\gamma\left(1 - \frac{u \cdot V}{c^2}\right)}$$

$$\underline{V}_{AB} = \underline{V}_A + \underline{V}_B\left[\frac{\underline{V}_A \cdot \underline{V}_B}{\underline{V}_B \cdot \underline{V}_B}(\gamma_B - 1) - \gamma_B\right] \Big/ \gamma_B\left(1 - \frac{V_A \cdot V_B}{c^2}\right) \quad \text{--- (1)}$$

$\because$ $\underline{V}_A \cdot \underline{V}_B = 0$, $\underline{V}_A \cdot \underline{V}_A = u^2$, $\underline{V}_B \cdot \underline{V}_B = V^2$

$\underline{V}_c \cdot \underline{V}_c = \omega^2$, $\underline{V}_A \cdot \underline{V}_c = V_c \cdot V_A = 0$

$\underline{V}_c \cdot \underline{V}_B = \underline{V}_B \cdot \underline{V}_c = 0$ and $\underline{V}_B \cdot \underline{V}_A = 0$

Also:

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\gamma_A = \frac{1}{\sqrt{1 - \frac{V_A \cdot V_A}{c^2}}} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\gamma_B = \frac{1}{\sqrt{1 - \frac{V_B \cdot V_B}{c^2}}} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\gamma_c = \frac{1}{\sqrt{1 - \frac{\underline{V}_c \cdot \underline{V}_c}{c^2}}} = \frac{1}{\sqrt{1 - \frac{\omega^2}{c^2}}}$$

So (1) $\Rightarrow$

$$\underline{V}_{AB} = \frac{\underline{V}_A - \underline{V}_B \gamma_B}{\gamma_B} = \frac{(u, 0, 0) - (0, -V, 0)\gamma_B}{\gamma_B}$$

$$\underline{V}_{AB} = \frac{(u, V\gamma_B, 0)}{\gamma_B} \quad \text{where } \gamma_B = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \left(\frac{u}{\gamma_B}, V, 0\right)$$

$$\underline{V}_{AC} = \frac{V_A + V_C\left[\frac{V_A \cdot V_C}{V_C \cdot V_C}(\gamma_C - 1) - \gamma_C\right]}{\gamma_C\left(1 - \frac{V_A \cdot V_C}{c^2}\right)}$$

$$= \frac{V_A - V_C\gamma_C}{\gamma_C}$$

$$= \frac{(u, 0, 0) - (0, 0, w)\gamma_C}{\gamma_C}$$

$$= \frac{(u, 0, -w\gamma_C)}{\gamma_C}$$

$$\underline{V}_{AC} = \left(\frac{u}{\gamma_C}, 0, -w\right) \quad \text{where } \gamma_C = \frac{1}{\sqrt{1 - \frac{w^2}{c^2}}}$$

$$\underline{V}_{BA} = \frac{\underline{V}_B + \underline{V}_A\left[\frac{V_B \cdot V_A}{V_A \cdot V_A}(\gamma_A - 1) - \gamma_A\right]}{\gamma_A\left(1 - \frac{V_B \cdot V_A}{c^2}\right)}$$

$$= \frac{\underline{V}_B - V_A\gamma_A}{\gamma_A}$$

$$= \frac{(0, -v, 0) - (u, 0, 0)\gamma_A}{\gamma_A}$$

$$\underline{V}_{BA} = \left(-u, -\frac{v}{\gamma_A}, 0\right) \quad \text{where } \gamma_A = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\underline{V}_{CA} = \underline{V}_c + \underline{V}_A\left[\frac{\underline{V}_c \cdot \underline{V}_A}{\underline{V}_A \cdot \underline{V}_A}(\gamma_A - 1) - \gamma_A\right] \Big/ \gamma_A\left(1 - \frac{V_c \cdot V_A}{c^2}\right)$$

$$= \frac{\underline{V}_c - \underline{V}_A \gamma_A}{\gamma_A}$$

$$\frac{(0, 0, W) - (u, 0, 0)\gamma_A}{\gamma_A}$$

$$\underline{V}_{CA} = \frac{(-u\gamma_A, 0, W)}{\gamma_A}$$

$$\underline{V}_{BC} = \underline{V}_B + \underline{V}_c\left[\frac{\underline{V}_B \cdot \underline{V}_c}{\underline{V}_c \cdot \underline{V}_c}(\gamma_c - 1) - \gamma_c\right] \Big/ \gamma_c\left(1 - \frac{V_B \cdot V_c}{c^2}\right)$$

$$= \frac{\underline{V}_B - \underline{V}_c \gamma_c}{\gamma_c} = \frac{(0, -v, 0) - (0, 0, W)\gamma_c}{\gamma_c}$$

$$= \frac{(0, -v, -W\gamma_c)}{\gamma_c} \quad \text{where } \gamma_c = \frac{1}{\sqrt{1 - \frac{W^2}{c^2}}}$$

$$\underline{V}_{CB} = \underline{V}_c + \underline{V}_B\left[\frac{\underline{V}_c \cdot \underline{V}_B}{\underline{V}_B \cdot \underline{V}_B}(\gamma_B - 1) - \gamma_B\right] \Big/ \gamma_B\left(1 - \frac{\underline{V}_c \cdot \underline{V}_B}{c^2}\right)$$

$$= \frac{\underline{V}_c - \underline{V}_B \gamma_B}{\gamma_B}$$

$$= \frac{(0, 0, W) - (0, -v, 0)\gamma_B}{\gamma_B}$$

$$= \frac{(0, V\gamma_B, W)}{\gamma_B}$$

$$= \left(0, V, \frac{W}{\gamma_B}\right)$$

$$\gamma_A^2 = \frac{1}{1-\frac{u^2}{c^2}} = \frac{1}{\frac{c^2-u^2}{c^2}} = \frac{c^2}{c^2-u^2}$$

$$\gamma_B^2 = \frac{1}{1-v^2/c^2} = \frac{c^2}{c^2-v^2}$$

$$\gamma_C^2 = \frac{1}{1-w^2/c^2} = \frac{c^2}{c^2-w^2}$$

$$|V_{AB}| = \sqrt{\frac{u^2+v^2\gamma_B^2+0}{\gamma_B^2}}$$

$$= \frac{\sqrt{u^2+v^2 \cdot \frac{c^2}{c^2-v^2}}}{\gamma_B}$$

$$= \frac{\sqrt{u^2(c^2-v^2)+v^2c^2}}{\sqrt{c^2-v^2} \cdot \frac{c}{\sqrt{c^2-v^2}}}$$

$$|V_{AB}| = \frac{\sqrt{u^2c^2-u^2v^2+v^2c^2}}{c} \quad (*)$$

$$|V_{BA}| = \sqrt{\frac{u^2 \gamma_A^2 + V^2 + 0}{\gamma_A^2}}$$

$$= \frac{\sqrt{u^2 \cdot \frac{c^2}{c^2-u^2} + V^2}}{\gamma_B}$$

$$= \frac{\sqrt{c^2u^2 + c^2V^2 - u^2V^2}}{\sqrt{c^2-u^2} \cdot \frac{c}{\sqrt{c^2-u^2}}}$$

$$= \frac{\sqrt{c^2u^2 + c^2V^2 - u^2V^2}}{c} \quad \text{— } (*')$$

from (*) and (*')

$$|V_{AB}| = |V_{BA}|$$

Now for direction

$$\theta_{AB} = \tan^{-1}\left(\frac{V\gamma_B}{u}\right)$$

$$\theta_{BA} = \tan^{-1}\left(\frac{-V}{-u\gamma_A}\right)$$

$\Rightarrow$ θ_{AB} is in the 1st quadrant and θ_{BA} is in the 3rd quadrant and also they are not equal

$\because \gamma_A \neq \gamma_B$

Hence the direction is not equal nor opposite. The difference of θ_{AB} and θ_{BA} is due to not same value of γ_A and γ_B.

Now if the velocity of B is (0, V, 0) then

$$\underline{V}_{AB} = \frac{V_A - V_B\gamma_B}{\gamma_B}$$

$$= \frac{(u,0,0) - (0,V,0)\gamma_B}{\gamma_B}$$

$$= \frac{(u, -V\gamma_B, 0)}{\gamma_B} \quad \text{where} \quad \gamma_B = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$$

and $$\underline{V}_{BA} = \frac{V_B - V_A\gamma_A}{\gamma_A}$$

$$= \frac{(0,V,0) - (u,0,0)\gamma_A}{\gamma_A}$$

$$\underline{V}_{BA} = \frac{(-u\gamma_A, V, 0)}{\gamma_A} \quad \text{where} \quad \gamma_A = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

Since there is only sign difference in V_{AB}, V_{BA} i.e. they have same magnitude.

Now

$$\theta_{AB} = \tan^{-1}\left(\frac{-V\gamma_B}{u}\right), \quad \theta_{AB} \text{ is the 4th quad.}$$

$$\theta_{BA} = \tan^{-1}\left(\frac{V}{-u\gamma_n}\right), \quad \theta_{BA} \text{ is in the 2nd quad.}$$

So the direction is not same and not opposite.

2001

Q24 An observer A sees another observer B as moving with a velocity $(u, v, 0)$ and B sees C moving with a velocity of $(u, 0, v)$. What is the speed of C relative to A.

Soln ~~An observe~~ $\underline{V}_{BA} = (u, v, 0)$

$\underline{V}_{CB} = (u, 0, v)$

$\underline{V}_{CA} = ?$

$$V_{CA} = \underline{V}_c + \underline{V}_A\left[\frac{\underline{V}_c \cdot \underline{V}_A}{V_A \cdot V_A}(\gamma_A - 1) - \gamma_A\right] \Big/ \gamma_A\left(1 - \frac{\underline{V}_c \cdot \underline{V}_A}{c^2}\right)$$

Now according to given situation

$$\underline{V}_{CA} = \frac{V_{CB} + V_{BA}\left[\frac{V_{CB}\cdot V_{BA}}{V_{BA}\cdot V_{BA}}(\gamma_{BA}-1)-\gamma_{BA}\right]}{\gamma_{BA}\left(1-\frac{V_{CB}\cdot V_{BA}}{c^2}\right)}$$

Now as

$$\underline{V}_{CB}\cdot \underline{V}_{BA} = u^2$$

$$V_{BA}\cdot V_{BA} = u^2+v^2$$

So

$$\underline{V}_{CA} = \underline{V}_{CB} + \underline{V}_{BA}\left[\frac{u^2}{u^2+v^2}(\gamma_{BA}-1)-\gamma_{BA}\right] \Big/ \gamma_{BA}\left(1-\frac{u^2}{c^2}\right)$$

$$= \frac{\underline{V}_{CB} + \underline{V}_{BA}\left[\frac{u^2\gamma_{BA}-u^2-u^2\gamma_{BA}-v^2\gamma_{BA}}{u^2+v^2}\right]}{\gamma_{BA}\left(1-\frac{u^2}{c^2}\right)}$$

$$= \underline{V}_{CB}+\underline{V}_{BA}\left(\frac{-u^2-v^2\gamma_{BA}}{u^2+v^2}\right)\Big/\gamma_{BA}\left(1-\frac{u^2}{c^2}\right)$$

$$= (u,0,v)+(u,v,0)\left(\frac{-u^2-v^2\gamma_{BA}}{u^2+v^2}\right)\Big/\gamma_{BA}\left(1-\frac{u^2}{c^2}\right)$$

$$= \frac{(u,0,v)+\left(\frac{-u^3-uv^2\gamma_{BA}}{u^2+v^2}, \frac{-u^2v-v^3\gamma_{BA}}{u^2+v^2}, 0\right)}{\gamma_{BA}\left(1-\frac{u^2}{c^2}\right)}$$

$$= \frac{\left(u+\frac{-u^3-uv^2\gamma_{BA}}{u^2+v^2}, \frac{-u^2v-v^3\gamma_{BA}}{u^2+v^2}, v\right)}{\gamma_{BA}\left(1-u^2/c^2\right)}$$

$$= \frac{\left(\frac{u^3+uv^2-u^3-uv^2\gamma_{BA}}{u^2+v^2}, \frac{-u^2v-v^3\gamma_{BA}}{u^2+v^2}, v\right)}{\gamma_{BA}\left(1-\frac{u^2}{c^2}\right)}$$

$$= \frac{\left(\frac{uv^2(1-\gamma_{BA}), -v(u^2+v^2\gamma_{BA})}{u^2+v^2}, v\right)}{\gamma_{BA}\left(1-\frac{u^2}{c^2}\right)}$$

Q25 Two observers A and B see a rocket taking off at distances d and d' at times t and t' respectively, having set their clocks at zero when they are coincident. If their relative motion is perpendicular to the direction of rocket, and A sees the motion of it speed as V, what will be the velocity of the rocket according to B.

Soln let u_{yA}, u_{yB} be component of velocity of rocket w.r.t A and B. Let u be the relative motion of A and B

Now we have to find out

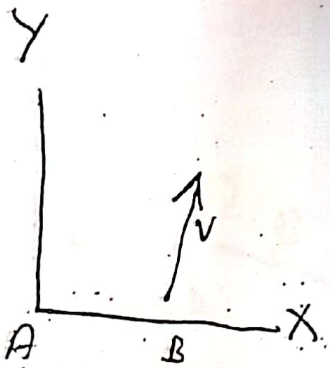
$$U_B = (U_{xB}, U_{yB}, U_{zB})$$

As we know that

$$U_y' = \frac{U_y\sqrt{1-V^2/c^2}}{1-\frac{U_x V}{c^2}}$$

also

$$U_x' = \frac{U_x - V}{1-\frac{U_x V}{c^2}}$$



$$\therefore U_{xB} = \frac{U_{xA} - V}{1-\frac{U_{xA} V}{c^2}}$$

$$U_{xA} = 0, \quad U_{yA} = 0 \text{ — } V$$

$$\therefore U_{xB} = \frac{0-V}{1-0} = -V$$

Similarly

$$U_{yB} = \frac{U_{yA}\sqrt{1-V^2/c^2}}{1-\frac{U_{xA}V}{c^2}}$$

$$= \frac{V\sqrt{1-V^2/c^2}}{1-0}$$

$$U_{yB} = V\sqrt{1-V^2/c^2}$$

$$U_{zB} = 0$$

Hence required velocity of Rocket according to B

is $U_B = \left(-v, v\sqrt{1-\frac{v^2}{c^2}}, 0\right)$

Q26 A particle is observed to have twice the mass it has at rest with what speed ~~mass~~ is it moving? relative to the observer? Another observer sees the particle at rest. If he looks at a meter rod hold by the 1st observer in the direction of motion. what length will be seen it has

Sol If m' is the observed mass and m is rest mass

Then $m' = \dfrac{m}{\sqrt{1-\frac{v^2}{c^2}}}$

Now observed mass is twice than the rest mass

i.e $m' = 2m$

$\therefore 2m = \dfrac{m}{\sqrt{1-\frac{v^2}{c^2}}}$

$2 = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$\Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \dfrac{1}{2}$

$\Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{4}$

$\frac{v^2}{c^2} = 1-\frac{1}{4}$

$$V = \frac{\sqrt{3}}{2} C = \frac{\sqrt{3}}{2} \times 3\times10^8$$

$$V = 2.6 \times 10^8 \, m/sec$$

If l' is the observed length and l is the rest length

$\therefore \; l = 1 \, m$

As we know that

$$l' = l\sqrt{1 - v^2/c^2}$$

$$l' = 1\left(\tfrac{1}{2}\right)$$

$$l' = 0.5 \, m$$

Q27 An object A moves with velocity $(u, 0, 0)$ relative to O, and another object B moves with a velocity $(0, v, 0)$ relative to O. What is the velocity of B relative to A? Show that the velocity of A relative to B is opposite to the velocity of B relative to A.

Sol'n

$$\underline{V}_{AO} = (u, 0, 0)$$

$$\underline{V}_{BO} = (0, v, 0)$$

Now we have to calculate V_{AB} and V_{BA}. As we know that

$$u' = \frac{\underline{u} + \underline{v}\left[\frac{u \cdot v}{\underline{v}\cdot\underline{v}}(\gamma-1) - \gamma\right]}{\gamma\left(1 - \frac{u \cdot v}{c^2}\right)}$$

As

$$\underline{V}_{AB} = \frac{\underline{V}_{A0} + V_{B0}\left[\frac{V_{A0}\cdot V_{B0}}{V_{B0}\cdot V_{B0}}(\gamma_B - 1) - \gamma_B\right]}{\gamma_B\left(1 - \frac{V_{A0}\cdot V_{B0}}{c^2}\right)}$$

Here $\underline{V}_{A0}\cdot\underline{V}_{B0} = 0$, $V_{A0}\cdot V_{A0} = u^2$

$V_{B0}\cdot V_{B0} = V^2$

$$\therefore \underline{V}_{AB} = \frac{(u,0,0) + (0,v,0)(-\gamma_B)}{\gamma_B}$$

$$= \frac{(u, -v\gamma_B, 0)}{\gamma_B} \quad \text{where, } \gamma_B = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\underline{V}_{AB} = \left(\frac{u}{\gamma_B}, -v, 0\right)$$

$$V_{BA} = \frac{\underline{V}_{B0} + \underline{V}_{A0}\left[\frac{V_{B0}\cdot V_{A0}}{V_{A0}\cdot V_{A0}}(\gamma_A - 1) - \gamma_A\right]}{\gamma_A\left(1 - \frac{V_{B0}\cdot V_{A0}}{c^2}\right)}$$

$$\underline{V}_{BA} = \frac{(0,v,0) + (u,0,0)(-\gamma_A)}{\gamma_A}$$

$$= \left(-u, \frac{v}{\gamma_A}, 0\right)$$

where $\gamma_A = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$

the velocity of $\underline{V}_{AB}$ and $\underline{V}_{BA}$ are opposite If

$$\underline{V}_{AB} = -\underline{V}_{BA}$$

It follows only If

$$\frac{u}{\gamma_B} = -u \quad , \quad \frac{v}{\gamma_A} = -v$$

$\Rightarrow$ ~~γ_B~~

Decay Process in High energy physics

we shall now Consider the dynamics of the decay of the particle D into two particles A and B.

Let m_d, m_a and m_b be the rest masses. E_d, E_a and E_b be the energies. $P(d)$, $P(a)$ and $P(b)$ be the three momenta of the Particles. then according to the laws of conservation of energies and momentum we have

$$E_d = E_a + E_b \quad \text{—— (1)}$$

$$P(d) = P(a) + P(b) \quad \text{—— (2)}$$

Suppose at the time of decay particle d is at rest.

$$\Rightarrow P(d) = 0$$

So Eq (2) ⇒

$$P(a) = -P(b) \quad \text{—— (3)}$$

i.e the momenta of the products of the decay are equal and opposite. Squaring equation (3) we have

$$[P(a)]^2 = [P(b)]^2 = P^2 \quad \text{—— (4)}$$

Since we know that

$$E^2 = m_0^2 c^4 + P^2 c^2$$

$$E_d^2 = m_d^2 c^4$$

$$\Rightarrow E_d = m_d c^2 \quad \text{—— (5)}$$

&

$$E_a^2 = m_a^2 c^4 + [P(a)]^2 c^2$$

$$E_a^2 = m_a^2 c^4 + P^2 c^2 \quad \text{—— (6)}$$

Similarly, $E_b^2 = m_b^2 c^4 + P^2 c^2$ —— (7)

Eq (6) − Eq (7) ⇒

$$E_a^2 - E_b^2 = (m_a^2 - m_b^2) c^4 \quad \text{—— (8)}$$

$$(E_a - E_b)(E_a + E_b) = (m_a^2 - m_b^2) c^4$$

from ① and ⑤

$$E_a + E_b = m_d c^2 \quad \text{--- (9)}$$

Subsituting $E_a + E_b$ in Equation ⑧

$$(E_a - E_b)\, m_d c^2 = (m_a^2 - m_b^2)\, c^4$$

$$\Rightarrow E_a - E_b = \frac{(m_a^2 - m_b^2)\, c^2}{m_d} \quad \text{--- (10)}$$

⑨ + ⑩ ⇒

$$E_a = \frac{m_d^2 + m_a^2 - m_b^2}{2 m_d} \quad \text{--- (11)}$$

This gives the total energy of the decay product A.

the total energy of the decay product B is obtained by Subtracting ⑩ from ⑨

$$E_b = \frac{m_d^2 + m_b^2 - m_a^2}{2 m_d} c^2 \quad \text{--- (12)}$$

Suppose next the particle decays in flight such that emerging particles A and B move in directions making an angle ϕ

with each other

Squaring equation (2) we have

$$P_{(d)}^2 = P_{(a)}^2 + P_{(b)}^2 + 2 P_{(a)} P_{(b)} \cos\phi \quad \text{(13)}$$

Now

$$E_d^2 = m_d^2 c^4 + P_{(d)}^2 c^2 \quad \text{(14)}$$

$$E_a^2 = m_a^2 c^4 + P_{(a)}^2 c^2 \quad \text{(15)}$$

$$E_b^2 = m_b^2 c^4 + P_{(b)}^2 c^2 \quad \text{(16)}$$

Subsituting the expression for $P_{(d)}^2$, $P_{(a)}^2$ and $P_{(b)}^2$ from equation (14), (15) and (16) in equation (13)

$$\frac{E_d^2 - m_d^2 c^4}{c^2} = \frac{E_a^2 - m_a^2 c^4}{c^2} + \frac{E_b^2 - m_b^2 c^4}{c^2} + 2 P_{(a)} P_{(b)} \cos\phi$$

$$E_d^2 - m_d^2 c^4 = E_a^2 - m_a^2 c^4 + E_b^2 - m_b^2 c^4 + 2 P_{(a)} P_{(b)} \cos\phi \cdot c^2$$

From equation (1)

$$E_d^2 = E_a^2 + E_b^2 + 2 E_a E_b$$

$$\Rightarrow P_{(a)} P_{(b)} \cos\phi \cdot c^2 = 2 E_a E_b + m_a^2 c^4 + m_b^2 c^4 - m_d^2 c^4$$

$$\cos\phi = \frac{(m_a^2 + m_b^2 - m_d^2)c^4 + 2E_a E_b}{2P(a)P(b)c^2}$$

this equation gives the angle between two products.

2987 1986

Q28 A π^+ meson of rest mass m_π decays into a muon of rest mass m_μ and neutrino of zero rest mass

i.e $\pi^+ \longrightarrow \mu^+ + \nu_\mu$

Assuming that π^+ is stationary at the time of its decay. Find the expression for K.E of muon and the corresponding neutrino.

Soln

$$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} c^2$$

$$E_\nu = \frac{m_\pi^2 + m_\nu^2}{2m_\pi} c^2$$

$$T_\mu = \{E_\mu - (m_\mu)c^2$$

$$= \left(\frac{m_\pi^2 + m_\mu^2}{2 m_\pi} - m_\mu \right) c^2$$

$$= \frac{\left(m_\pi^2 + m_\mu^2 - 2 m_\pi m_\mu \right)}{2 m_\pi} c^2$$

$$T_\mu = \frac{(m_\pi - m_\mu)^2 c^2}{2 m_\pi}$$

$$T_\gamma = \{ E_\gamma - (m_\gamma) c^2$$

$$= \left(\frac{m_\pi^2 + m_\gamma^2}{2 m_\pi} - m_\gamma \right) c^2$$

$$T_\gamma = \frac{(m_\pi - m_\gamma)^2}{2 m_\pi} c^2$$

Q 29 A Λ^0 (Hyperon) of rest mass m_Λ decays into two massive particles. A proton p of mass m_p and meson of mass m_π according to the relation

$$\Lambda^0 \longrightarrow \bar{\pi} + p$$

Assuming that the Hyperon is stationary at the

time of its decay. Find the expression for the total and K.E of proton and π^-.

Sol'n

As we know that

$$E_p = \frac{m_\Lambda^2 + m_p^2 - m_\pi^2}{2 m_\Lambda} c^2$$

$$E_\pi = \frac{m_\Lambda^2 + m_\pi^2 - m_p^2}{2 m_\Lambda} c^2$$

The Corresponding K.E T_p and T_π are given by

$$T_p = E_p - m_p c^2$$

$$= \left(\frac{m_\Lambda^2 + m_p^2 - m_\pi^2}{2 m_\Lambda} - m_p \right) c^2$$

$$= \left(\frac{m_\Lambda^2 + m_p^2 - m_\pi^2 - 2 m_p m_\Lambda}{2 m_\Lambda} \right) c^2$$

$$= \frac{(m_\Lambda - m_p)^2 - m_\pi^2}{2 m_\Lambda} c^2$$

& $$T_\pi = \frac{(m_\Lambda - m_\pi)^2 - m_p^2}{2 m_\Lambda} c^2$$

Ans

Q Suppose that a π^0 meson of mass m_π decays into two photons of different frequencies. i.e $\pi^0 \rightarrow \gamma + \gamma$ Calculate the angle between the photons

Sol'n The decay of π^0 meson of rest mass m_π and momentum P into two photons of different frequencies ν_1 and ν_2.

The rest mass of the decay particle is m_π.

The rest masses of the products are zero. The energies of the photon produced in this process are

$$E_a = h\nu_1$$
$$E_b = h\nu_2$$

The corresponding momenta's are

$$P_a = h\nu_1/c$$
$$P_b = h\nu_2/c$$

As we know that

$$\cos\phi = \frac{(m_a^2 + m_b^2 - m_\pi^2)c^4 + 2E_aE_b}{2P(a)P(b)c^2}$$

$$\Rightarrow \cos\phi = \frac{(0+0-m_\pi^2)c^4 + 2(h\nu_1)(h\nu_2)}{2\frac{h\nu_1}{c} \times \frac{h\nu_2}{c} \times c^2}$$

$$= \frac{-m_\pi^2 c^4}{2h^2\nu_1\nu_2} + \frac{2h^2\nu_1\nu_2}{2h^2\nu_1\nu_2}$$

$$= 1 - \frac{m_\pi^2 c^4}{2h^2\nu_1\nu_2} \quad \text{Ans}$$

Rapidity :-

According to the theorey of relativity. the resultant w of the two callinear velocities u and v is given by

$$w = \frac{u+v}{1+\frac{uv}{c^2}} \quad \text{--- (1)}$$

which is different from classical resultant for the addition of callinear velocities.

We can however define a quantity called 'Rapidity' or pseuodo velocity which